Calculus II

MATH 152 Course Notes

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Preface

This booklet contains the note templates for courses *Math 150/151 - Calculus I* at Simon Fraser University. Students are expected to use this booklet during each lecture by following along with the instructor, filling in the details in the blanks provided.

Definitions and theorems appear in highlighted boxes.

Next to some examples you'll see [link to applet]. The link will take you to an online interactive applet to accompany the example - just like the ones used by your instructor in the lecture. The link above will take you to the following url [Mul22] containing all the applets:

```
http://www.sfu.ca/~jtmulhol/calculus-applets/html/appletsforcalculus.html
```

Try it now.

Next to some section headings you'll notice a QR code. They look like the image on the right.
Each one provides a link to a webpage (could be a youtube video, or access to online Sage code). For example this one takes you to the Wikipedia page which explains what a QR code is. Use a QR code scanner on your phone or tablet and it will quickly take you off to the webpage. The app "Red Laser" is a good QR code scanner which is available for free (iphone, android, windows phone).

(wikipedia)

http://en.wikipedia.org/wiki/QR_code

We offer a special thank you to Keshav Mukunda for his many contributions to these notes.

No project such as this can be free from errors and incompleteness. We will be grateful to everyone who points out any typos, incorrect statements, or sends any other suggestion on how to improve this manuscript.

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January 13, 2023

Greek Alphabet

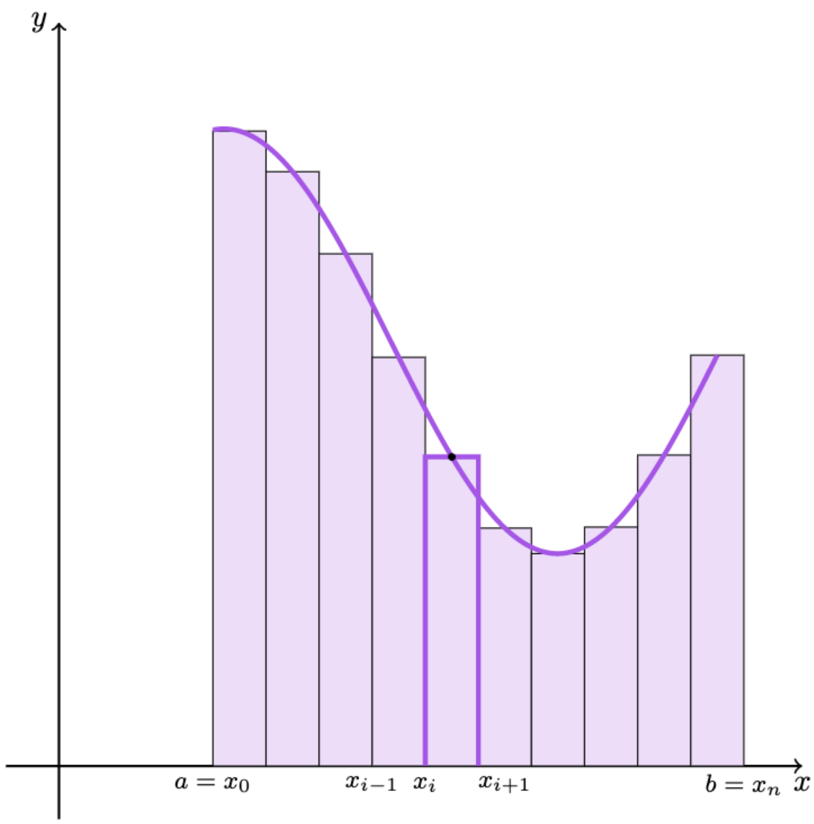
lower case	capital	name	pronunciation	lower case	capital	name	pronunciation
α	A	alpha	(al-fah)	ν	N	nu	(new)
β	B	beta	(bay-tah)	ξ	Ξ	xi	(zie)
γ	Γ	gamma	(gam-ah)	o	O	omicron	(om-e-cron)
δ	Δ	delta	(del-ta)	π	Π	pi	(pie)
ε	E	epsilon	(ep-si-lon)	ρ	P	rho	(roe)
ζ	Z	zeta	(zay-tah)	σ	Σ	sigma	(sig-mah)
η	H	eta	(ay-tah)	τ	T	tau	(taw)
θ	Θ	theta	(thay-tah)	υ	Υ	upsilon	(up-si-lon)
ι	I	iota	(eye-o-tah)	ϕ	Φ	phi	(fie)
κ	K	kappa	(cap-pah)	χ	X	chi	(kie)
λ	Λ	lambda	(lamb-dah)	ψ	Ψ	psi	(si)
μ	M	mu	(mew)	ω	Ω	omega	(oh-may-gah)

Part One: Introduction to the Integral

1 Integrals **13**
1.1 Areas and Distances
1.2 The Definite Integral
1.3 The Fundamental Theorem of Calculus
1.4 The Net Change Theorem
1.5 The Substitution Rule

2 Applications of Integration **47**
2.1 Areas Between Curves
2.2 Areas in Polar Coordinates
2.3 Volumes
2.4 Volumes by Cylindrical Shells

1. Integrals



In this chapter we lay down the foundations for this course. We introduce the two motivating problems for integral calculus: the area problem, and the distance problem. We then define the integral and discover the connection between integration and differentiation.

1.1 Areas and Distances

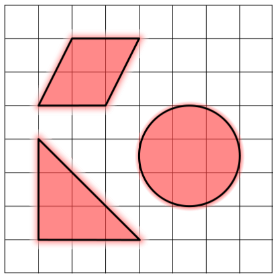
(This lecture corresponds to Section 5.1 of Stewart's *Calculus*.)

One can never know for sure what a deserted ***area*** *looks like.*
(George Carlin, American stand-up Comedian, Actor and Author, 1937-2008)

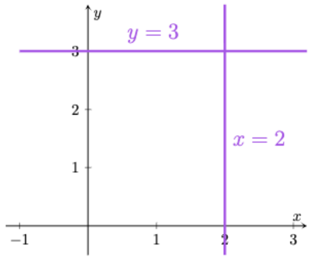
BIG Question. What is the meaning of the word **area**?
Vocabulary. Cambridge dictionary:

area *noun*

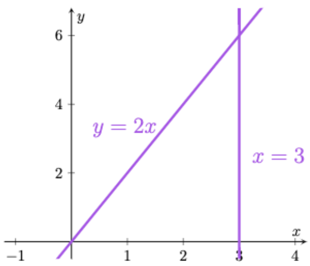
(a) a particular part of a place, piece of land or country;
(b) the size of a flat surface calculated by multiplying its length by its width;
(c) a subject or activity, or a part of it.
(d) (Wikipedia) - Area is a physical quantity expressing the size of a part of a surface.



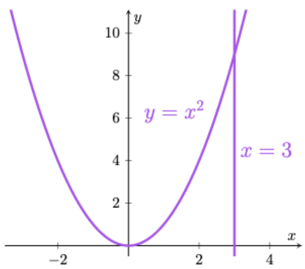
Example 1.1 Find the area of the region in the coordinate plane bounded by the coordinate axes and lines $x = 2$ and $y = 3$.



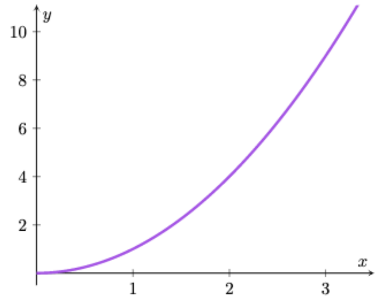
Example 1.2 Find the area of the region in the coordinate plane bounded by the x-axis and lines $y = 2x$ and $x = 3$.



Example 1.3 Find the area of the region in the coordinate plane bounded by the x-axis and lines $y = x^2$ and $x = 3$.



Example 1.4 *Estimate* the area of the region in the coordinate plane bounded by the x-axis and curves $y = x^2$ and $x = 3$.



Example 1.5 **(Over- and under-estimates.)** In the previous example, show that

$$\lim_{n\to\infty} R_n = 9 \quad \text{and} \quad \lim_{n\to\infty} L_n = 9.$$

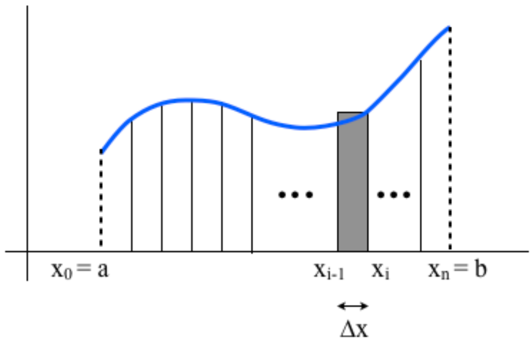
A more general formulation.
Ingredients: A function f that is continuous on a closed interval $[a,b]$.
Let $n \in \mathbb{N}$, and define $\Delta x = \frac{b-a}{n}$.
Let

$$\begin{aligned} x_0 &= a \\ x_1 &= a+\Delta x \\ x_2 &= a+2\Delta x \\ x_3 &= a+3\Delta x \\ \vdots & \\ x_n &= a+n\Delta x = b. \end{aligned}$$

Define

$$R_n = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \ldots + f(x_n)\cdot\Delta x.$$

("R" stands for "right-hand", since we are using the right hand endpoints of the little rectangles.)



Definition 1.1.1 — Area. The **area** A of the region S that lies under the graph of the continuous function f over and interval $[a,b]$ is the limit of the sum of the areas of approximating rectangles R_n. That is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1) + f(x_2) + \ldots + f(x_n)]\,\Delta x.$$

The more compact **sigma notation** can be used to write this as

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i) \right) \Delta x.$$

Example 1.6 Find the area under the graph of $f(x) = 100 - 3x^2$ from $x = 1$ to $x = 5$.

From the definition of area, we have $A = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i) \right) \Delta x.$

Distance Problem. Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

Reminder **distance = velocity · time**

Additional Notes:

1.2 The Definite Integral

(This lecture corresponds to Section 5.2 of Stewart's *Calculus*.)

> *After years of finding mathematics easy, I finally reached integral calculus and came up against a barrier. I realized that this was as far as I could go, and to this day I have never successfully gone beyond it in any but the most superficial way.*
>
> (Isaac Asimov, Russian-born American author and biochemist, best known for his works of science fiction, 1920-1992)

Definition 1.2.1 — The Definite Integral. Suppose f is a continuous function defined on the closed interval $[a,b]$, we divide $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. Let

$$x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_n = b$$

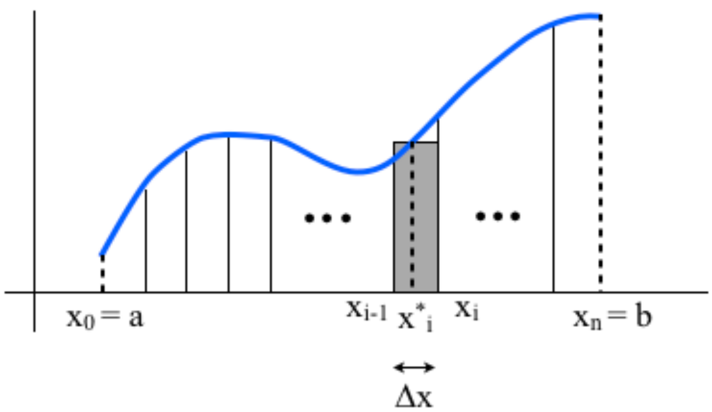
be the end points of these subintervals. Let

$$x_1^*, x_2^*, \ldots, x_n^*$$

be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$.

Then the **definite integral of f from a to b** is written as $\displaystyle\int_a^b f(x)dx$, and is defined as follows:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



The definite integral: some terminology

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- $\int$ is the *integral sign*
- $f(x)$ is the *integrand*
- a and b are the *limits of integration:*
  - a - *lower limit*
  - b - *upper limit*
- The procedure of calculating an integral is called *integration*.
- $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*
  (named after the German mathematician Bernhard Riemann,1826-1866)

Four Facts.

(a) If $f(x) > 0$ on $[a,b]$ then $\int_a^b f(x)dx > 0$.
If $f(x) < 0$ on $[a,b]$ then $\int_a^b f(x)dx < 0$.

(b) For a general function f,

$$\int_a^b f(x)dx = \text{(signed area of the region) = (area above } x\text{-axis) - (area below } x\text{-axis)}$$

(c) For every $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

$$\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

for every $n > N$ and every choice of $x_1^*, x_2^*, \ldots, x_n^*$.

(d) Let f be continuous on $[a,b]$ and let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be any partition of $[a,b]$. Let $\Delta x_i = x_i - x_{i-1}$, and suppose $\max \Delta x_i$ approaches 0 as n tends to infinity. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Some facts you just have to know.[1]

(a)

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(b)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(c)

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(d)

$$\sum_{i=1}^{n} c = cn$$

(e)

$$\sum_{i=1}^{n} (ca_i) = c\sum_{i=1}^{n} a_i$$

(f)

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

[1] For visual proofs of (a) and (b) see [Gol02]: Goldoni, G. (2002) *A visual proof for the sum of the first n squares and for the sum of the first n factorials of order two*. The Mathematical Intelligencer 24 (4): 67-69. You can access the Mathematical Intelligencer through the SFU Library web site: `http://cufts2.lib.sfu.ca/CJDB/BVAS/journal/150620`.

Example 1.7 Evaluate
$$\int_0^2 (x^2 - x)dx.$$

Example 1.8 Express the limit
$$\lim_{n\to\infty} \sum_{i=1}^{n} (1 + x_i)\cos x_i \Delta x$$
as a definite integral on the interval $[\pi, 2\pi]$.

Example 1.9 Prove
$$\int_0^2 \sqrt{4 - x^2}dx = \pi.$$

Theorem 1.2.1 — Choosing a good sample point . . . . **Midpoint Rule.** To approximate an integral it is usually better to choose x_i^* to be the midpoint $\bar{x}_i$ of the interval $[x_{i-1}, x_i]$:

$$\int_a^b f(x)dx \quad \approx \quad \sum_{i=1}^{n} f(\bar{x}_i)\Delta x \quad = \quad \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)\right]$$

Recall the midpoint of an interval $[x_{i-1}, x_i]$ is given by $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

Example 1.10 Use the Midpoint Rule with $n = 4$ to approximate the integral $\displaystyle\int_1^5 \frac{dx}{x^2}$.

Theorem 1.2.2 — Two Special Properties of the Integral.

(a) If $a > b$ then
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

(b) If $a = b$ then
$$\int_a^b f(x)dx = 0.$$

Some More Properties of the Integral.

(a) If c is a constant, then $\displaystyle\int_a^b c dx = c(b-a)$

(b) $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(c) If c is a constant, then $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$

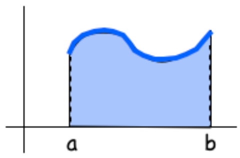
(d) $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Example 1.11 Evaluate $\displaystyle\int_0^3 \left(2x - 3\sqrt{9-x^2}\right) dx.$

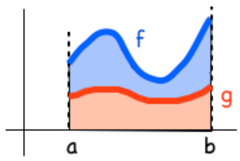
Example 1.12 Evaluate $\int_0^3 f(x)dx$ if $f(x) = \begin{cases} 1-x & \text{if} \quad x \in [0,1] \\ -\sqrt{1-(x-2)^2} & \text{if} \quad x \in (1,3] \end{cases}$

More Properties of the definite integral.

(a) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

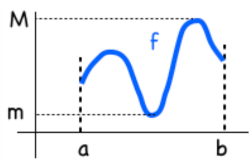


(b) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.



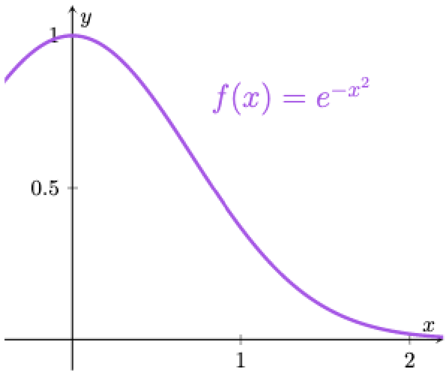
(c) If m and M are constants, and $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$



Example 1.13 Prove

$$\frac{1}{e^4} \le \int_1^2 e^{-x^2} dx \le \frac{1}{e}$$



Example 1.14 (a) If f is continuous on $[a,b]$, show that

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx.$$

(b) Show that if f is continuous on $[0,2\pi]$ then

$$\left| \int_0^{2\pi} f(x)\sin 2x dx \right| \le \int_0^{2\pi} |f(x)|dx.$$

Additional Notes:

1.3 The Fundamental Theorem of Calculus

(This lecture corresponds to Section 5.3 of Stewart's *Calculus*.)

All of my fundamental principles that were instilled in me in my home, from my childhood, are still with me.

(Hakeem Abdul Olajuwon, a former NBA player,1963-)

Motivating Problem Does every continuous function f have an antiderivative? That is, does there exist a function F such that
$$F'(x) = f(x)?$$

Motivating Problem What is the antiderivative of $f(x) = \dfrac{\sin x}{x}$?

Theorem 1.3.1 — The Fundamental Theorem of Calculus, Part 1. If f is a continuous on $[a,b]$, then the function g defined by
$$g(x) = \int_a^x f(t)dt,\ a \le x \le b$$
is continuous on $[a,b]$ and differentiable on (a,b), and
$$g'(x) = f(x).$$

Example 1.15 Apply the Fundamental Theorem of Calculus, Part 1, to find the derivative of the following functions:

(a) $g(x) = \int_1^x \frac{\sin t}{t} dt$

(b) $g(x) = \int_0^{x^2} \sin t \, dt$

(c) $g(x) = \int_0^{h(x)} f(t) \, dt$

(d) $g(x) = \int_{-3x}^{e^x} \ln(1+t^2) dt$

Theorem 1.3.2 — The Fundamental Theorem of Calculus, Part 2. If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f. That is, a function such that $F' = f$.

Example 1.16 Evaluate the following integrals:

(a) $\int_0^1 x dx$

(b) $\int_2^3 e^x dx$

(c) $\int_0^\pi \sin x \, dx$

(d) $\int_0^1 \frac{dx}{1+x^2}$

Example 1.17 A Piecewise Example. Let

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \\ 2 - x & \text{if} \quad 1 < x \leq 2 \\ 0 & \text{if} \quad x > 2 \end{cases}$$

and let $g(x) = \int_0^x f(t)dt$.

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f differentiable? Where is g differentiable?

Additional Notes:

1.4 The Net Change Theorem

(This lecture corresponds to Section 5.4 of Stewart's *Calculus*.)

At the end of some indefinite distance there was always a confused spot, into which her dream died.

(Gustave Flauber, French novelist, 1821-1880)

Reminder **The Fundamental Theorem of Calculus, Part 2:**
If f is continuous on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is a function such that $F' = f$.

Motivating Problem So, to be able to evaluate an integral, we need a way to find any antiderivative F of the given function f . How do we find antiderivatives?

A new name for an old idea...

Definition 1.4.1 The symbol $\int f(x)dx$ is called an **indefinite integral**, and it represents an antiderivative of f. That is,

$$\int f(x)dx = F(x) \text{ means } F'(x) = f(x)$$

Warning! It could be confusing: The notation $\int f(x)dx$ is used to represent

- the **set** of all antiderivatives of f

$$\int f(x)dx = \{F : F' = f\}$$

- a single **function** that is an antiderivative of f.

Integrals you should know:

$$\int cf(x)dx = c\int f(x)dx \qquad \int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{dx}{x^2+1} = \tan^{-1} x + C \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

Example 1.18 Find the following indefinite integrals:

(a) $\displaystyle\int x^{-2/3}dx$

(b) $\displaystyle\int t^2(3-4t^5)dt$

(c) $\displaystyle\int (u-1)(u^2+3)du$

(d) $\displaystyle\int (4e^v-\sec^2 v)dv$

(e) $\displaystyle\int \frac{\cos z}{1-\cos^2 z}dz$

Theorem 1.4.1 — The Net Change Theorem. The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example 1.19 If $f(x)$ is the slope of a hiking trail at a distance of x miles from the start of the trail, what does $\int_2^4 f(x)\,dx$ represent?

Example 1.20 — Linear Motion of a Particle. A particle is moving along a line with the acceleration (in m/s^2) $a(t) = 2t + 3$ and the initial velocity $v(0) = -4$ m/s with $0 \le t \le 3$. Find

(a) the velocity at time t,

(b) the distance traveled during the given time interval.

Additional Notes:

1.5 The Substitution Rule

(This lecture corresponds to Section 5.5 of Stewart's *Calculus*.)

Persuasion is often more effectual than force.

(Aesop, Greek fabulist, 6th century BC)

Motivating Problem Find

$$\int -2xe^{-x^2}\,dx$$

Hint. What if we think of the "dx" above as a differential? If $u = e^{-x^2}$, what is the differential du?

Theorem 1.5.1 — The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Notes:

(a) This rule can be proved using the Chain Rule for differentiation. In this sense, it is a reversal of the Chain Rule.

(b) The substitution rule says that we can work with "dx" and "du" that appear after the $\int$ symbols as if they were differentials.

Example 1.21 Find the following indefinite integrals:

(a) $\int x^2(x^3+5)^9\,dx$

(b) $\displaystyle\int \frac{dt}{\sqrt{3-5t}}$

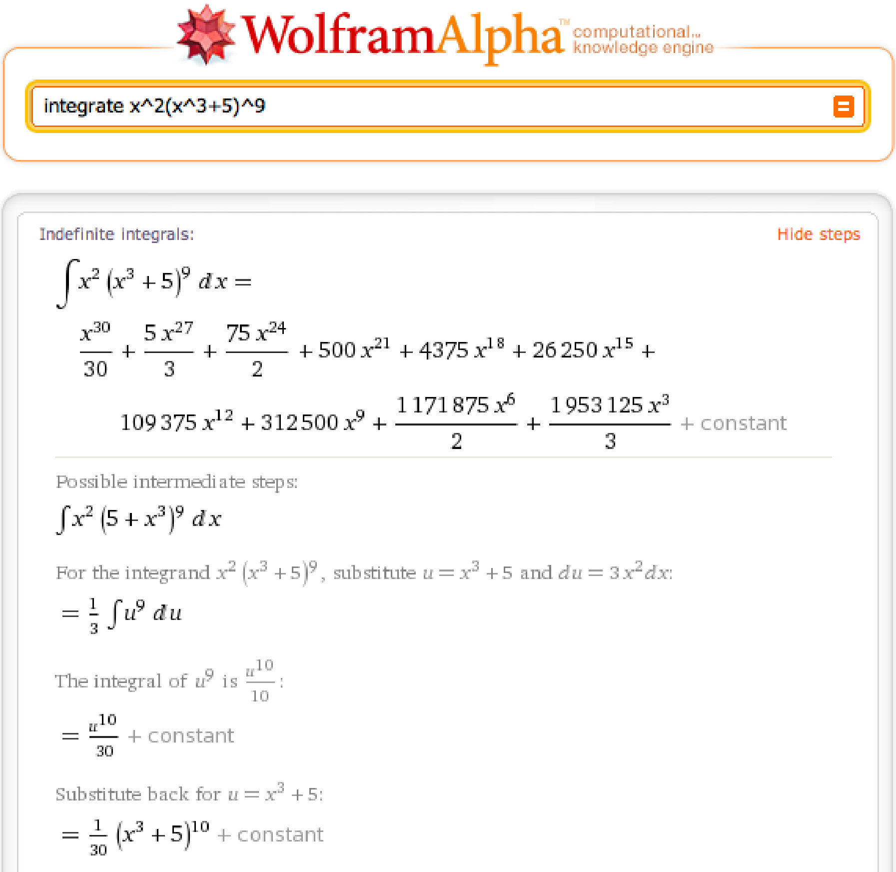
(c) $\displaystyle\int \sin 3t \; dt$

(d) $\displaystyle\int \frac{du}{u(\ln u)^2}$

(e) $\displaystyle\int \frac{\sin(\pi/v)}{v^2} \; dv$

(f) $\displaystyle\int \frac{z^2}{\sqrt{1-z}}\,dz$

Computers are ideal for computing integrals, and Wolfram|Alpha (www.wolframalpha.com) gives you easy access to this computing power. Use it as a tool to help you study.
But be warned: you still have to understand how to do these computations yourself, since Wolfram|Alpha won't be with you for quizzes and exams.



Theorem 1.5.2 — Substitution Rule for Definite Integrals. If g' is continuous on $[a,b]$ and if f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Notes:

(a) When we make the substitution $u = g(x)$, then the interval $[a,b]$ on the x-axis becomes the interval $[g(a), g(b)]$ on the u-axis.

(b) Writing

$$\int_a^b f(g(x))g'(x)\,dx \;=\; \int_a^b f(u)\,du \;=\; \int_{g(a)}^{g(b)} f(u)\,du$$

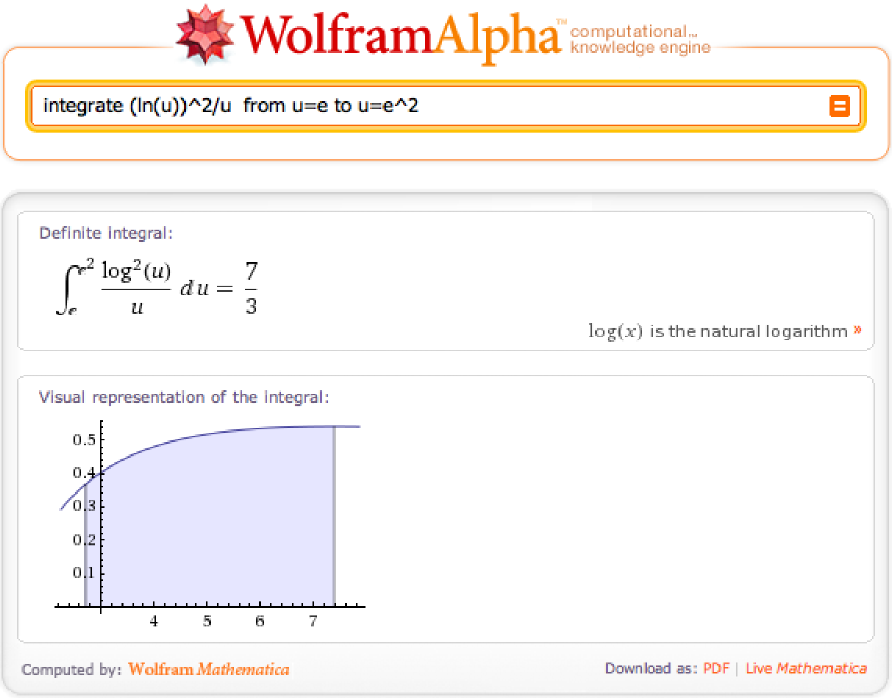
would **NOT** be right.
Make the substitution **AND** change the limits of integration at the same time!

Example 1.22 Evaluate the following definite integrals:

(a) $\displaystyle\int_{\pi}^{2\pi} \cos 3t\,dt$

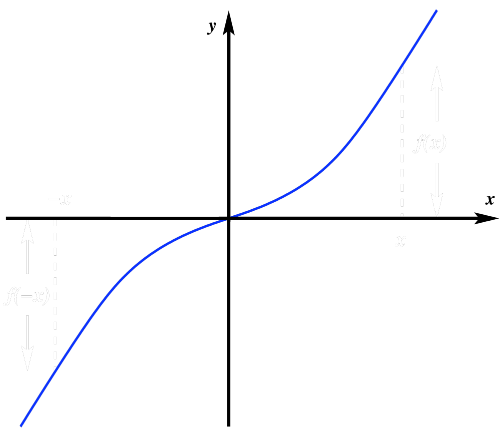
(b) $\displaystyle\int_e^{e^2} \frac{(\ln u)^2 du}{u}$

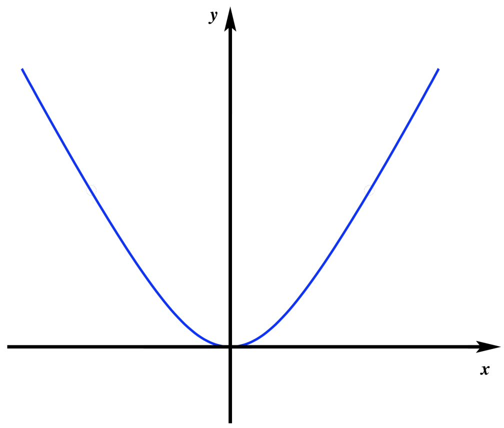
Again, use Wolfram|Alpha to check your answer.



Even or Odd? Let $a > 0$ and let f be continuous on $[-a, a]$.

- If f is **odd** then
$$\int_{-a}^{a} f(x)dx = 0$$
- If f is **even** then
$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$





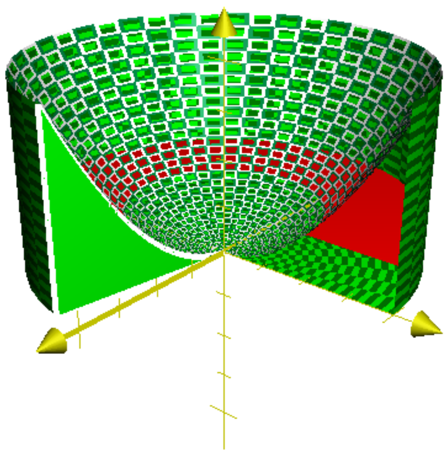
Example 1.23 Evaluate the following definite integrals:

(a) $\displaystyle\int_{-3}^{3}(2x^4+3x^2+4)dx$

(b) $\displaystyle\int_{-e}^{e}\frac{e^{-u^2}\sin u\,du}{u^2+10}$

Additional Notes:

2. Applications of Integration



In the last chapter we learned how to compute the area between a curve and the x-axis, this gave rise to the *integral*.
In this chapter the topics we will cover are:

- the area between two curves,
- the area bounded by a curve given in polar coordinates,
- the disk and washer methods for computing the volume of a solid.

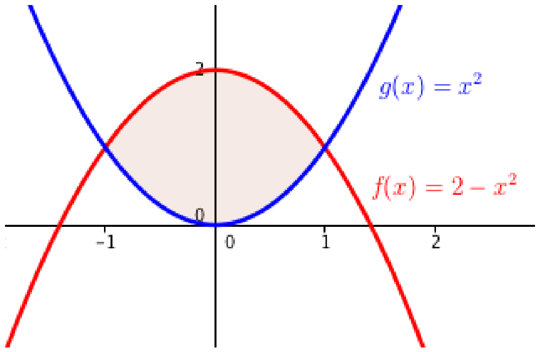
2.1 Areas Between Curves

(This lecture corresponds to Section 6.1 of Stewart's *Calculus*.)

> *Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.*
>
> (David Hilbert, German mathematician, 1862-1943)

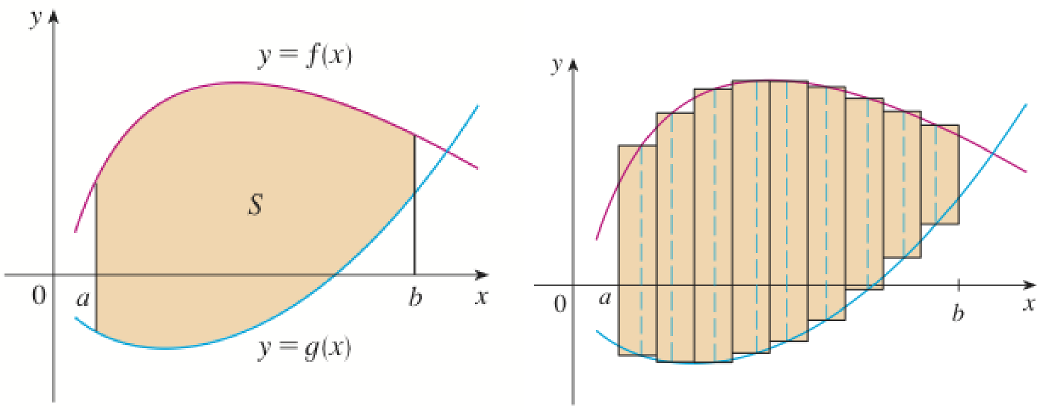
Motivating Problem Find the area bounded by parabolas

$$y = 2 - x^2 \text{ and } y = x^2.$$



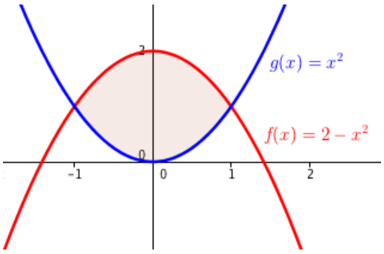
Theorem 2.1.1 — Area Between Curves. Suppose f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a,b]$. The area A bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, is given by

$$A = \int_a^b f(x) - g(x)\, dx.$$

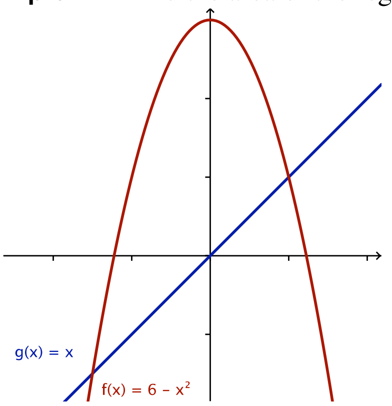


Example 2.1 Find the area bounded by parabolas

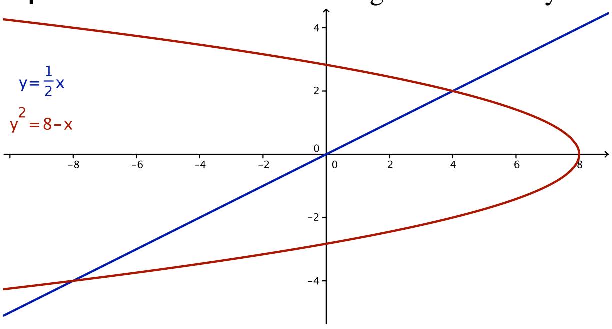
$$y = 2 - x^2 \text{ and } y = x^2.$$



Example 2.2 Find the area of the region bounded by the line $y = x$ and the parabola $y = 6 - x^2$.



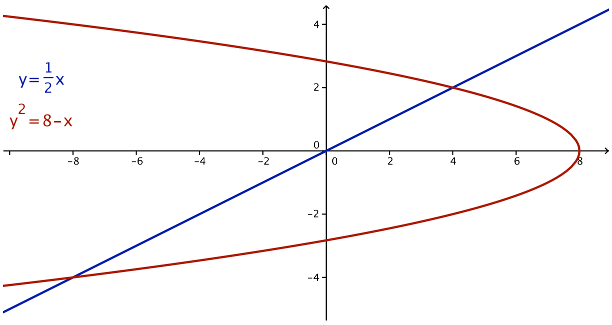
Example 2.3 Find the area of the region bounded by the line $y = x/2$ and the parabola $y^2 = 8 - x$.



Theorem 2.1.2 — Doing this area calculation along the y-axis.... Suppose the area A is bounded by the curves $x = f(y)$, $x = g(y)$, and the lines $y = c$, $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for all $y \in [c,d]$. Then the area is given by

$$A = \int_c^d [f(y) - g(y)]dy.$$

Example 2.4 Find the area of the region bounded by the line $y = x/2$ and the parabola $y^2 = 8 - x$.



Additional Notes:

2.2 Areas in Polar Coordinates

(This lecture corresponds to Section 10.4 of Stewart's *Calculus*.)

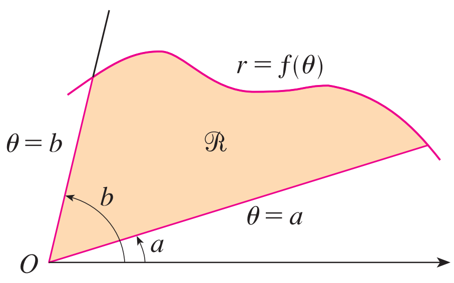
These [equations] are your friends. Use them, know them, love them.
(Donna Pierce, American astrophysicist, 1975-)

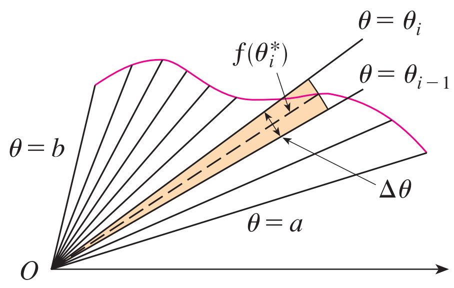
Motivating Problem Sketch the curve and find the area that it encloses:

$$r = 1 + \cos\theta.$$

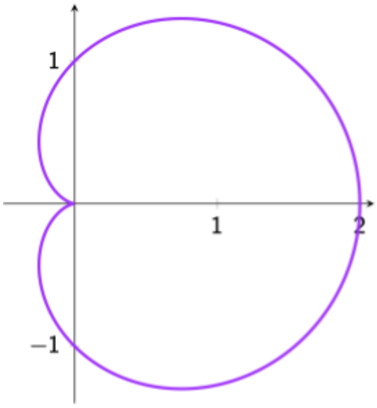
Theorem 2.2.1 — Area bounded by polar curves. The area of a polar region $\mathscr{R}$ bounded by the curve $r = f(\theta)$, for $\theta \in [a,b]$, is given by

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta.$$

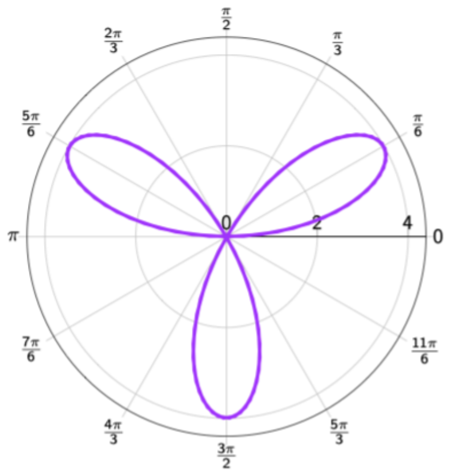




Example 2.5 Find the area enclosed by $r = 1 + \cos\theta$.



Example 2.6 Find the area of the region enclosed by the 3-leaved rose $r = 4\sin(3\theta)$.



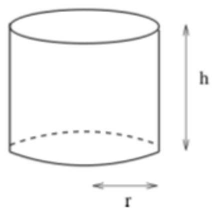
Additional Notes:

2.3 Volumes

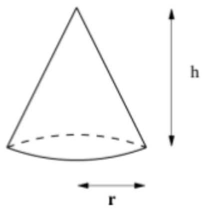
(This lecture corresponds to Section 6.2 of Stewart's *Calculus*.)

> *I shall now recall to mind that the motion of the heavenly bodies is circular, since the motion appropriate to a sphere is rotation in a circle.*
> (Nicolaus Copernicus, mathematician, astronomer, jurist, physician, classical scholar, governor, administrator, diplomat, economist, and soldier, 1473-1543)

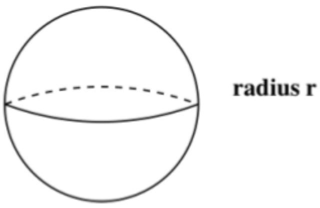
Recall some classic volume formulas:



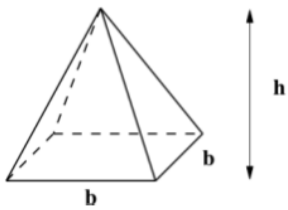
Right circular cylinder, $V = \pi r^2 h$



Cone, $V = \frac{1}{3}\pi r^2 h$



Sphere, $V = \frac{4}{3}\pi r^3$

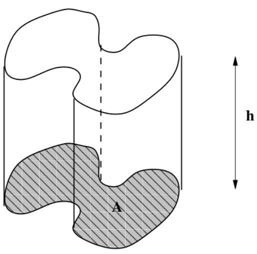


Square pyramid, $V = \frac{1}{3}b^2 h$

Motivating Problem How do we prove these formulas? Moreover, how do we define the volume of a solid object?

Definition 2.3.1 — Definition of Volume... simple beginnings.

(i) The volume of a general cylinder with cross sectional area A and height h is defined to be Ah.
(ii) The volume of a general solid is defined using integrals (calculus).

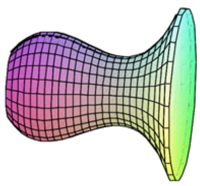


$$V = Ah$$

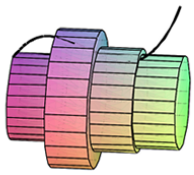
Surprisingly, it turns out (by *Cavalieri's principle*) that these cross-sectional "area slices" can be rearranged and still give the same total volume.

Definition of Volume . . . the technique.

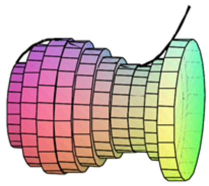
Problem: Find the volume of this solid.



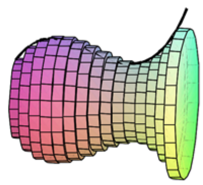
Approximate by 4 cylinders.



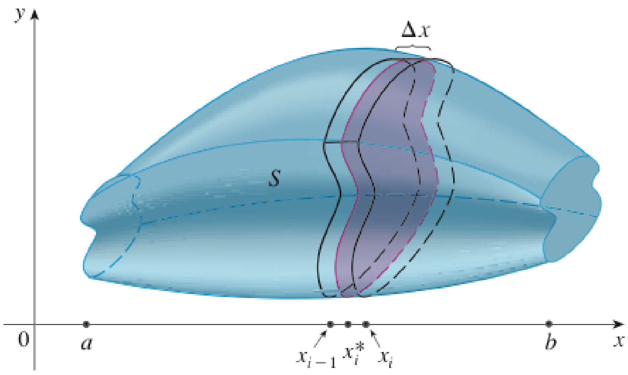
Approximate by 10 cylinders.

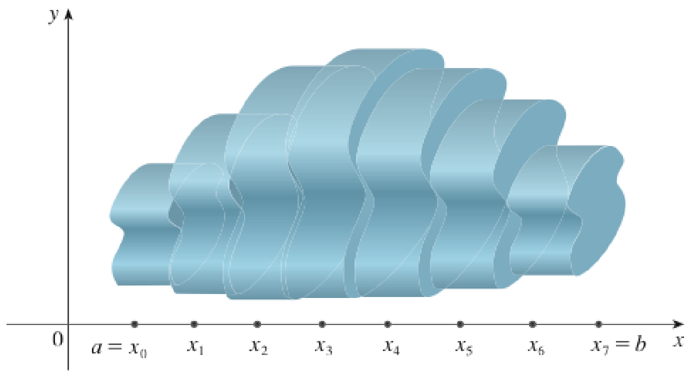


Approximate by 15 cylinders.



Computing the volume of a general solid S.

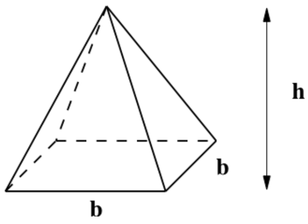


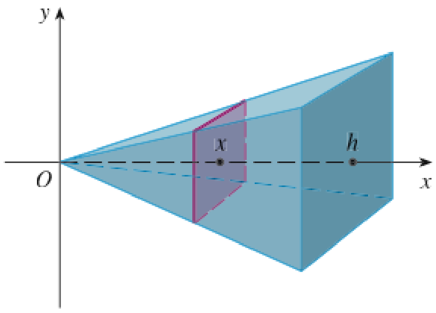


Definition 2.3.2 — Definition of Volume. Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \int_a^b A(x)dx.$$

Example 2.7 Find the volume of a pyramid whose base is a square with side b and whose height is h.

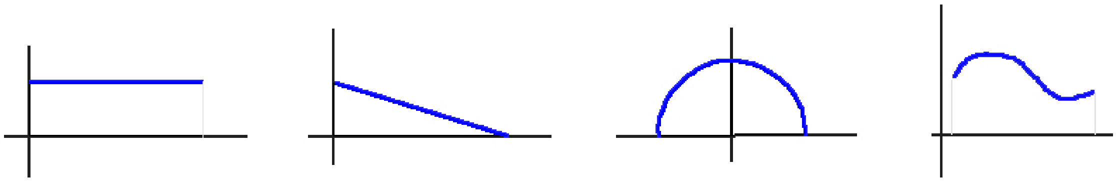




Definition 2.3.3 — Solid of Revolution. A solid of revolution is a solid (volume) obtained by revolving a region (or area) in the plane about a line.

In this case the cross-sections are disks or annuli (a.k.a disks or washers), so the volume formula $V = \int_a^b A(x)\, dx$ is known as the **washer method**.

Example 2.8 Some regions in the plane are shown below. Draw the resulting solid if these regions are rotated about the x-axis?



Example 2.9 Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = \sin x, \quad x = \frac{\pi}{2}, \text{ and } y = 0$$

about the x-axis.

Example 2.10 Find the volume of the solid obtained by rotating the region bounded by the curves

$$y = \sqrt{x}, \quad y = 1, \text{ and } x = 0$$

about the y-axis.

Example 2.11 Find the volume of the solid obtained by rotating the region R, which is enclosed by the curves $y = x$ and $y = x^3$ in the first quadrant, about the line

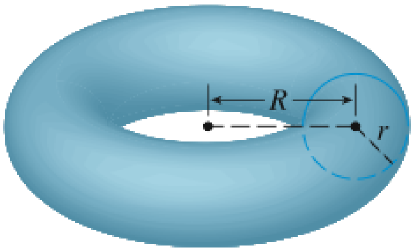
(a) $y = 3$

(b) $x = 2$

Example 2.12

(a) Set up an integral for the volume of a **torus** with inner radius r and outer radius R.

(b) By interpreting the integral as an area, find the volume of the torus.



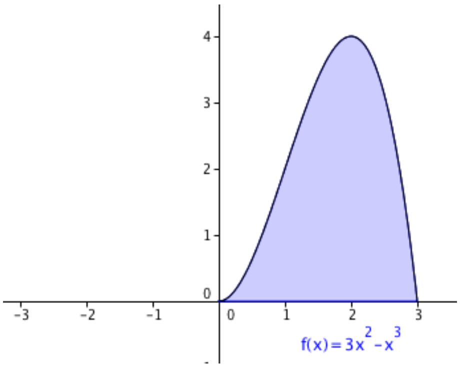
Additional Notes:

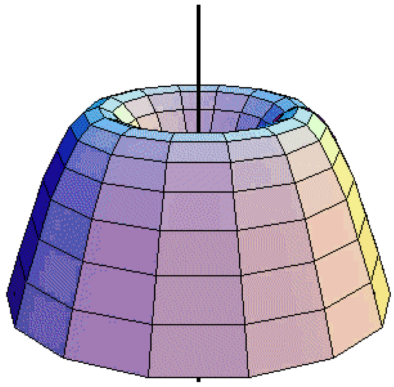
2.4 Volumes by Cylindrical Shells

(This lecture corresponds to Section 6.3 of Stewart's *Calculus*.)

> *Your pain is the breaking of the shell that encloses your understanding.*
> (Kahlil Gibran, Lebanese born American philosophical essayist, novelist and poet, 1883-1931)

Motivating Problem Consider the region in the xy-plane bounded by the curves $y = 3x^2 - x^3$ and $y = 0$. Imagine this region rotated about the y-axis. How do we find the volume of the resulting solid?

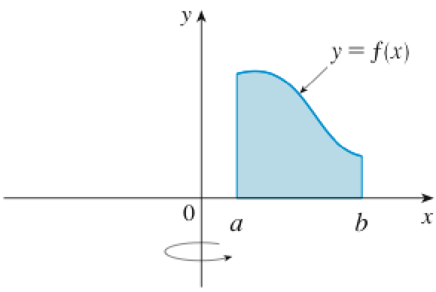


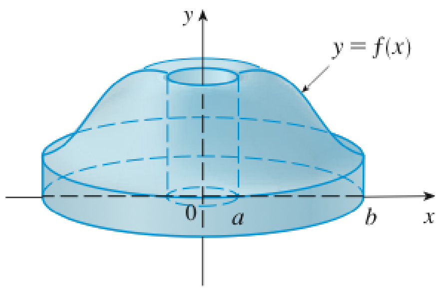


Exercise your imagination!
Let $0 \leq a < b$ and let a function f be continuous on $[a,b]$ with $f(x) \geq 0$. Let R be the region bounded by

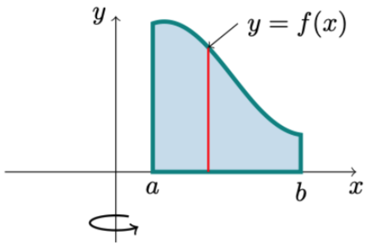
$$y = f(x),\ y = 0,\ x = a,\ \text{and } x = b.$$

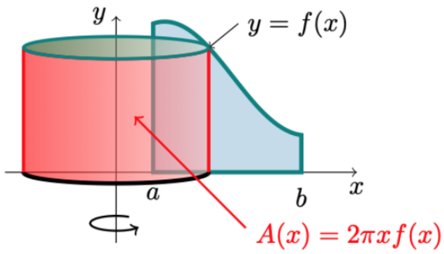
If we rotate R about the y-axis, we get a solid volume S.





Next, take an $x \in [a,b]$. Let L_x be the line segment inside the region R, between the points $(x,0)$ and $(x, f(x))$. Imagine that L_x is colored red. Now rotate L_x about the y-axis. Do this slowly so that you can see how a red cylinder with the radius x and the height $f(x)$ emerges. This is your cylindrical shell, called C_x.
The shell C_x is made of "skin" only. To calculate its surface we cut it along the line segment L_x and then flatten it to obtain a rectangle with the width $2\pi x$ and the height $f(x)$. Thus the surface of C_x equals $A_x = 2\pi x f(x)$.





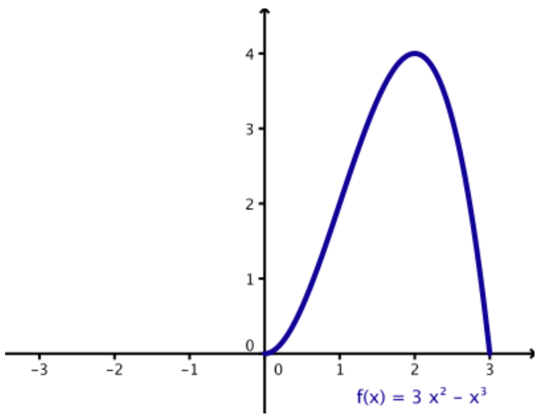
Almost there...
Note that each point of the solid S belongs to only one cylindrical shell C_x, for some $x \in [a,b]$. So we can imagine that S is obtained by gluing all cylindrical shells together. Each cylindrical shell contributes its surface (or "skin"!) to the volume of S, or, in other words, the volume is the "sum" of all surfaces. Each $x \in [a,b]$ gives one shell C_x with a surface area A_x, and so the "sum" of all of them is given by

$$V = \int_a^b A_x \, dx = 2\pi \int_a^b x f(x) \, dx.$$

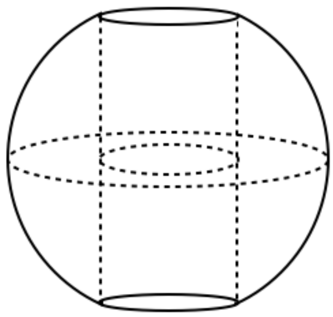
This is known as the **cylindrical shells method** (or simply the **shell method**) for computing the volume of a solid of revolution.

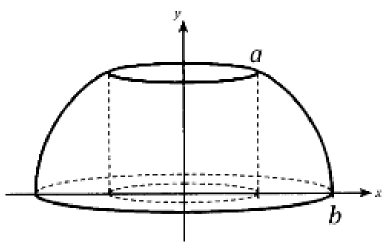
Example 2.13 Find the volume of the solid obtained by rotating about the y-axis the region bounded by curves

$$y = 3x^2 - x^3 \text{ and } y = 0.$$



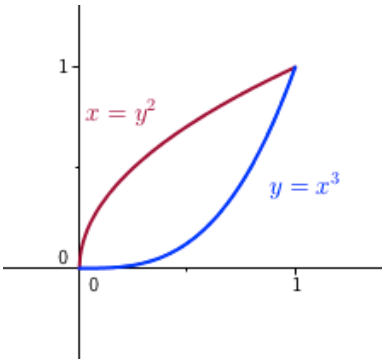
Example 2.14 Find the volume of the solid that remains after you bore a circular hole of radius a through the center of a solid sphere of radius $b > a$.



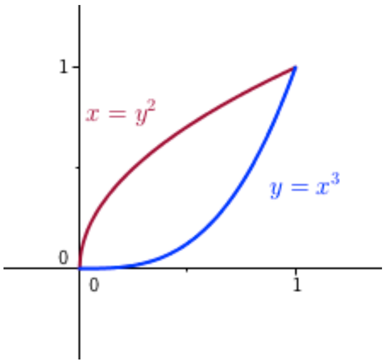


Example 2.15 Consider the region in the first quadrant bounded by the curves $y^2 = x$ and $y = x^3$. Use the method of cylindrical shells to compute the volume of the solid obtained by revolving this region around

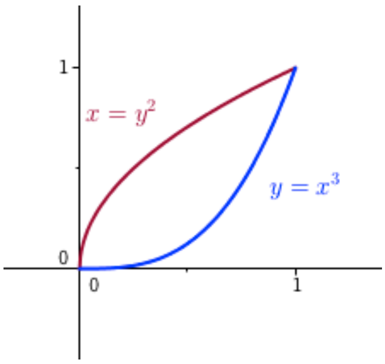
(a) y-axis.



(b) x-axis.



(c) line $x = 1$.



Summary: A general guideline for which method to use is the following:

- If the area section (strip) is **parallel** to the axis of rotation, use the **shell method**.
- If the area section (strip) is **perpendicular** to the axis of rotation, use the **washer method**.

Additional Notes:

Part Two: Integration Techniques and Applications

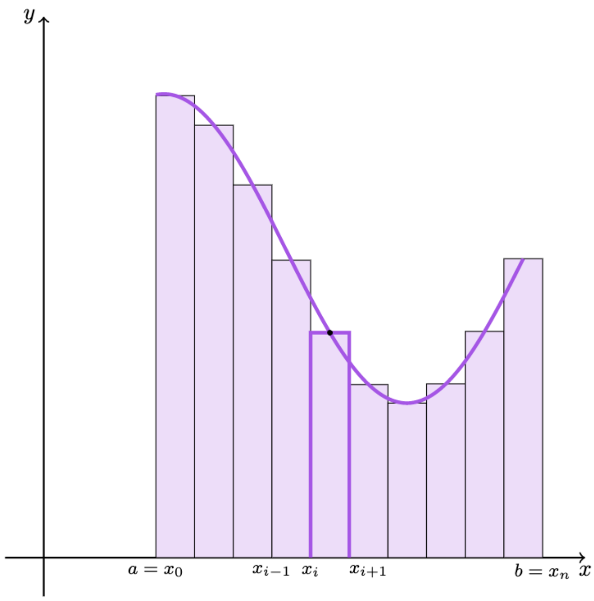
3 Techniques of Integration and Applications 69

3.1 Integration By Parts
3.2 Trigonometric Integrals
3.3 Trigonometric Substitutions
3.4 Integration of Rational Functions by Partial Fractions
3.5 Strategy for Integration
3.6 Approximate Integration
3.7 Improper Integrals

4 Further Applications of Integration . . . 109

4.1 Arc Length
4.2 Area of a Surface of Revolution
4.3 Calculus with Parametric Curves

3. Techniques of Integration and Applications



In this chapter we develop techniques for computing integrals (antiderivatives).

Topics we will cover are:

- integration by parts (i.e. undoing the product rule from differentiation),
- trigonometric integrals,
- substitutions with trigonometric functions,
- integration of rational functions by partial fractions,
- approximation of integrals,
- improper integrals.

3.1 Integration By Parts

(This lecture corresponds to Section 7.1 of Stewart's *Calculus*.)

> *Warning: this material is for a mature calculus audience.*
> *Disclaimer* on the web page *The absolutely outrageous CALCULUS IS COOL webpage* by Jochen Denzler, http://www.math.utk.edu/∼denzler/CalculusND/index.html
> (Jochen Denzler, German-born mathematician, 1963-)

Motivating Problem Integrate

$$\int xe^x dx.$$

Theorem 3.1.1 — Integration By Parts. Let f and g be differentiable functions. Then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Here is an easier way to remember this: for $u = f(x)$ and $v = g(x)$

$$\int udv = uv - \int vdu.$$

Example 3.1 Integrate $\int xe^x dx$.

Example 3.2 Integrate

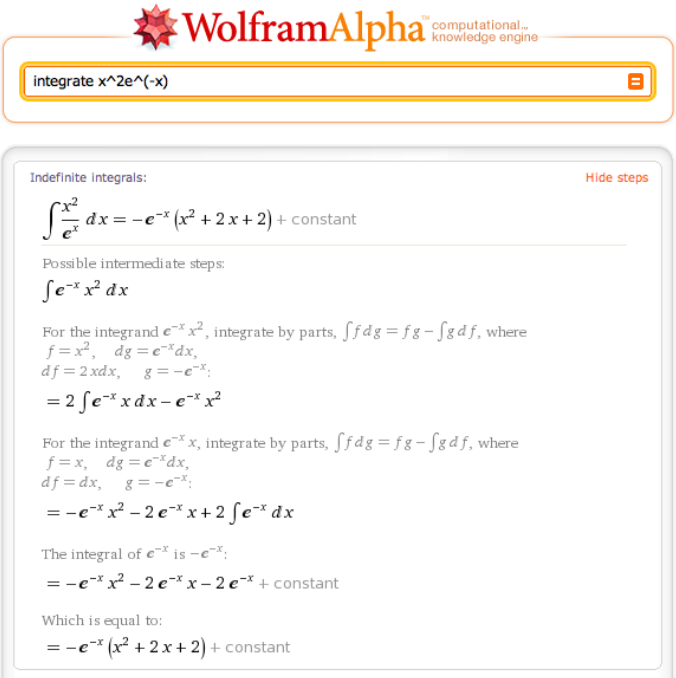
(a) $\int \ln x \, dx$

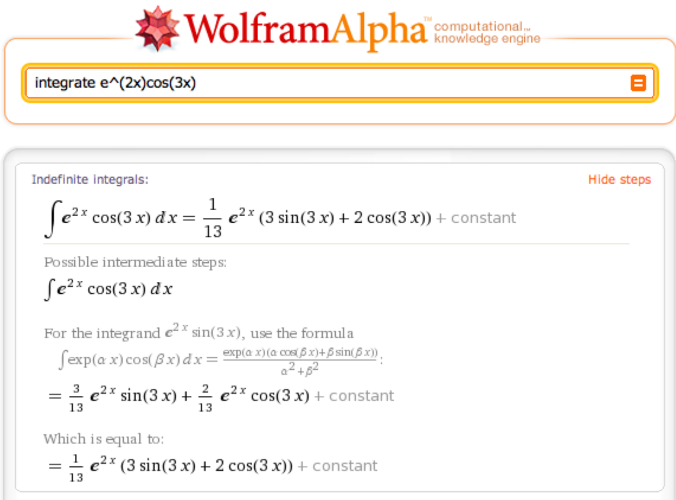
(b) $\int \arcsin x \, dx$

(c) $\int x^2 e^{-x} \, dx$

(d) $\int e^{2x} \cos 3x \, dx$

Check answers with Wolfram|Alpha:





Example 3.3 (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x \, dx$

(c) Use parts (a) and (b) to evaluate $\int \cos^4 x \, dx$

Example 3.4 Evaluate

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)\, dx.$$

Additional Notes:

3.2 Trigonometric Integrals

(This lecture corresponds to Section 7.2 of Stewart's *Calculus*.)

> *Today, I am giving two exams...one in trig and the other in honesty. I hope you will pass them both. If you must fail one, fail trig. There are many good people in the world who can't pass trig, but there are no good people who cannot pass the exam of honesty.*
>
> (Madison Sarratt, Dean and then Vice-Chancellor at Vanderbilt University, 1891-1978)

Example 3.5 Integrate the following:

(a) $\displaystyle\int \sin^2(3x)\,dx =$

(b) $\displaystyle\int \cot^2(3x)\,dx =$

Products of Sines and Cosines.

To evaluate $\int \sin^n x \cos^m x dx$, there are only two possibilities:

(a) At least one of the numbers n and m is **odd**.

Example 3.6 $\int \sin^3 x \cos^2 x \, dx =$

(b) Both n and m are **even**.

Example 3.7 $\int \sin^2 x \cos^2 x \, dx =$

Example 3.8 Integrate $\int \cos^5 x \, dx$

Example 3.9 Integrating Other Trig Functions: Tangent, Cotangent, Secant, and Cosecant.

(a) $\int \tan x \, dx =$

(b) $\int \cot x \, dx =$

(c) $\int \sec x \, dx =$

(d) $\int \csc x \, dx =$

Additional Notes:

3.3 Trigonometric Substitutions

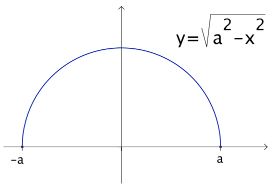
(This lecture corresponds to Section 7.3 of Stewart's *Calculus*.)

There is no harm in patience, and no profit in lamentation.

(Abu Bakr, The First Caliph, 573-634)

Here our goal is to use trig functions to try to simplify the integrand, hopefully converting it to one that is easier to integrate.

Motivating Problem Assuming that $|x| \leq a$, evaluate



$$\int \sqrt{a^2 - x^2}\, dx.$$

Integration by Substitution (using Trigonometric Functions

If the integral involves	then substitute	and use the identity
$a^2 - u^2$	$u = a \sin\theta$	$1 - \sin^2\theta \;=\; \cos^2\theta$
$a^2 + u^2$	$u = a \tan\theta$	$1 + \tan^2\theta \;=\; \sec^2\theta$
$u^2 - a^2$	$u = a \sec\theta$	$\sec^2\theta - 1 \;=\; \tan^2\theta$

Example 3.10 Integrate

$$\int \sqrt{1 - x^2}\, dx, \quad \text{assuming } |x| < 1$$

Additional Notes:

3.4 Integration of Rational Functions by Partial Fractions

(This lecture corresponds to Section 7.4 of Stewart's *Calculus*.)

It does not matter how slowly you go so long as you do not stop.
(Confucius, Chinese Philosopher, 551-479 BC)

Motivating Problem Evaluate
$$\int \frac{x-1}{x^2-5x+6}dx.$$

Algorithm — Integrating Rational Functions.
Problem. Evaluate $\int \frac{P(x)}{Q(x)}dx$, where P and Q are polynomials.

If $\deg P \geq \deg Q$ then (by long division) there are polynomials $q(x)$ and $r(x)$ such that
$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$
and either $r(x)$ is identically 0 or $\deg r < \deg Q$. The polynomial q is the quotient and r the remainder produced by the long division process.

If $r(x) = 0$, then $\frac{P(x)}{Q(x)}$ is really just a polynomial, so we can ignore that case here.

Now $\int \frac{P(x)}{Q(x)}dx = \int q(x)dx + \int \frac{r(x)}{Q(x)}dx.$

We can easily integrate the polynomial q, so the general problem reduces to the problem of integrating a rational function $\frac{r(x)}{Q(x)}$ with $\deg r < \deg Q$.

So, for the purposes of investigating how to integrate a rational function we can suppose $f(x) = \frac{P(x)}{Q(x)}$ with $\deg P(x) < \deg Q(x)$.

Theorem 3.4.1 — Fact About Every Polynomial Q. Q can be factored as a product of linear factors (i.e. of the form $ax+b$)

and / or

irreducible quadratic forms (i.e. of the form ax^2+bx+c, where $b^2-4ac<0$).

Our strategy to integrate the rational function $f(x)$ is as follows:

- Factor $Q(x)$ into linear and irreducible quadratic factors
- Write $f(x)$ as a sum of **partial fractions**, where each fraction is of the form

$$\frac{K}{(ax+b)^s} \quad \text{or} \quad \frac{Lx+M}{(ax^2+bx+c)^t}.$$

- Integrate each partial fraction in the sum.

Question. How do we find K, L, and M?

Let's look at some examples.

Example 3.11 Integrate

(a) $\displaystyle\int \frac{4x^2-3x-4}{x^3+x^2-2x}dx$

(b) $\displaystyle\int \frac{x^3-4x-1}{x(x-1)^3}dx$

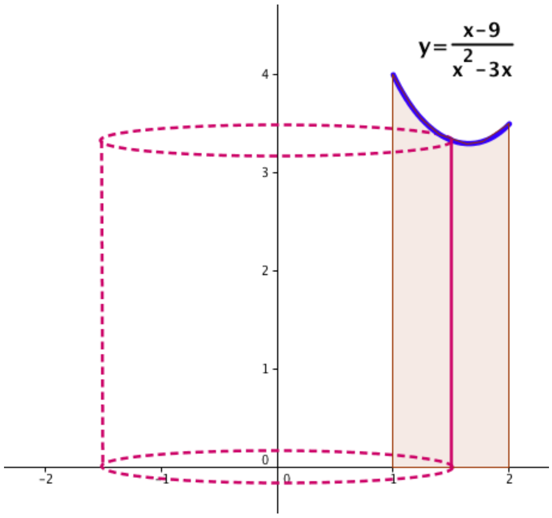
(c) $\displaystyle\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}dx$

(d) $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$

Example 3.12 Find the volume of the solid obtained by revolving the region R between the curve

$$y = \frac{x-9}{x^2-3x}$$

and the x-axis over the interval $1 \leq x \leq 2$, around the y-axis.



The steps to integrate a rational function f - A Technical Look

Suppose $f(x) = \frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$.

- **Step 1:** First factor $Q(x)$ into its linear and irreducible quadratic pieces. If there are n distinct linear factors and m distinct quadratic factors, then

$$Q(x) = (a_1x+b_1)^{r_1}\dots(a_nx+b_n)^{r_n}(c_1x^2+d_1x+e_1)^{s_1}\dots(c_mx^2+d_mx+e_m)^{s_m}$$

- **Step 2:** The $f(x)$ can be written as a sum of **partial fractions** as follows

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{A_{1,1}}{a_1x+b_1}+\frac{A_{1,2}}{(a_1x+b_1)^2}+\dots+\frac{A_{1,r_1}}{(a_1x+b_1)^{r_1}}+ \\
&\vdots \\
&+ \frac{A_{n,1}}{a_nx+b_n}+\frac{A_{n,2}}{(a_nx+b_n)^2}+\dots+\frac{A_{n,r_n}}{(a_nx+b_n)^{r_n}}+ \\
&+ \frac{B_{1,1}x+C_{1,1}}{c_1x^2+d_1x+e_1}+\frac{B_{1,2}x+C_{1,2}}{(c_1x^2+d_1x+e_1)^2}+\dots+\frac{B_{1,s_1}x+C_{1,s_1}}{(c_1x^2+d_1x+e_1)^{s_1}}+ \\
&\vdots \\
&+ \frac{B_{m,1}x+C_{m,1}}{c_mx^2+d_mx+e_m}+\frac{B_{m,2}x+C_{m,2}}{(c_mx^2+d_mx+e_m)^2}+\dots+\frac{B_{m,s_m}x+C_{m,s_m}}{(c_mx^2+d_mx+e_m)^{s_m}}
\end{aligned}$$

- **Step 3:** Integrate each partial fraction in the sum.

Additional Notes:

3.5 Strategy for Integration

(This lecture corresponds to Section 7.5 of Stewart's *Calculus*.)

> *A math student's best friend is BOB (the Back Of the Book), but remember that BOB doesn't come to school on test days.*
>
> (Joshua Folb, High School Teacher, Winchester, Virginia)

Table of Integration Formulas: Constants of integration have been omitted.

You should know this table!

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}, (n \neq -1) \qquad \int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x \qquad \int a^x\,dx = \frac{a^x}{\ln a}$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x \qquad \int \csc^2 x\,dx = -\cot x$$

$$\int \sec x \tan x\,dx = \sec x \qquad \int \csc x \cot x\,dx = -\csc x$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \csc x\,dx = \ln|\csc x - \cot x|$$

$$\int \tan x\,dx = \ln|\sec x| \qquad \int \cot x\,dx = \ln|\sin x|$$

$$\int \sinh x\,dx = \cosh x \qquad \int \cosh x\,dx = \sinh x$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

Example 3.13 Integrate

(a) $\int \frac{x+4}{x^3+x} dx$

(b) $\int_0^{\pi/2} \sin^4 x \cos^3 x dx$

(c) $\int e^x \sin(2x) dx$

Example 3.14 Integrate

(a) $\displaystyle\int_1^e \frac{\ln x}{x} dx$

(b) $\displaystyle\int \cos^2(5x) dx$

(c) $\displaystyle\int x^3 \ln x dx$

(d) $\displaystyle\int x \sec(x^2) \tan(x^2) dx$

(e) $\displaystyle\int \frac{3x+1}{x(x+1)} dx$

Example 3.15 Integrate

(a) $\displaystyle\int x^2(\ln x)^2 dx$

(b) $\displaystyle\int_0^{\pi/2} \cos^3 x \sin(2x) dx$

(c) $\displaystyle\int \frac{3}{x^{-1/2}(x^{3/2}-x^{1/2})} dx$

(d) $\displaystyle\int \frac{\sqrt{x^2-1}}{x} dx$ hint: Use the substitution $x = \sec\theta$.

(e) $\displaystyle\int_0^3 \frac{dx}{x^2-3x-4}$

Example 3.16 Integrate

(a) $\displaystyle\int x^5 e^{-x^3}\,dx$

(b) $\displaystyle\int_1^5 \sqrt{-x^2+6x-5}\,dx$

(c) $\displaystyle\int \frac{\sqrt{1+x}}{\sqrt{1-x}}dx$

(d) $\displaystyle\int \frac{\cos x}{4-\sin^2 x}dx$

(e) $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ hint: Use the substitution $x=\tan\theta$.

Additional Notes:

3.6 Approximate Integration

All exact science is dominated by the idea of approximation.

(Bertrand Russell, English Logician and Philosopher 1872-1970)

And now that you don't have to be perfect, you can be good.

(John Steinbeck, American Author, 1902-1968)

Motivating Problem Evaluate $\int_0^1 e^{-x^2}\,dx$.

Reminder If f is continuous on $[a,b]$ and if $[a,b]$ is divided into n subintervals

$$[a = x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n = b]$$

of equal length $\Delta x = \frac{b-a}{n}$ then

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where x_i^* is any point in $[x_{i-1}, x_i]$.

Various ways of choosing the sample points x_i^*:

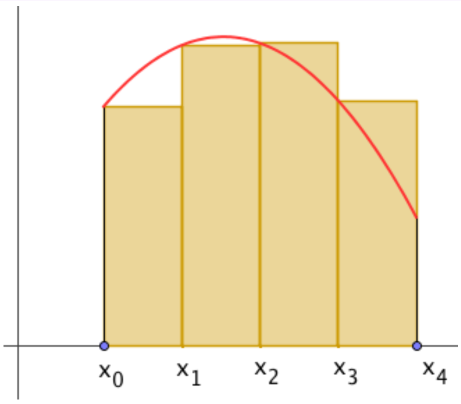
Endpoint Approximation.

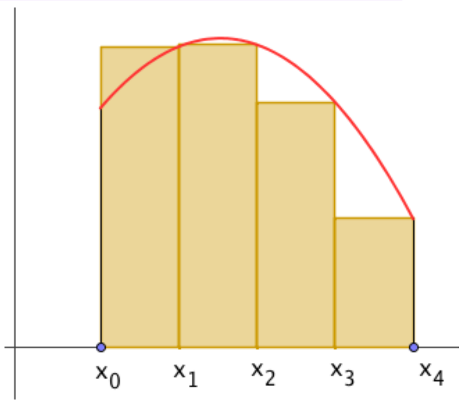
The left-point approximation L_n and the right-point approximation R_n to $\int_a^b f(x)dx$ with $\Delta x = \frac{b-a}{n}$ are

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

and

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$





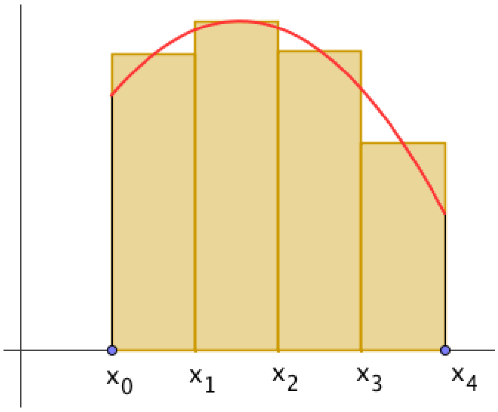
Midpoint Approximation.

The midpoint approximation M_n with $\Delta x = \frac{b-a}{n}$ is

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$

where

$$\bar{x}_i = \frac{x_{i-1}+x_i}{2}.$$



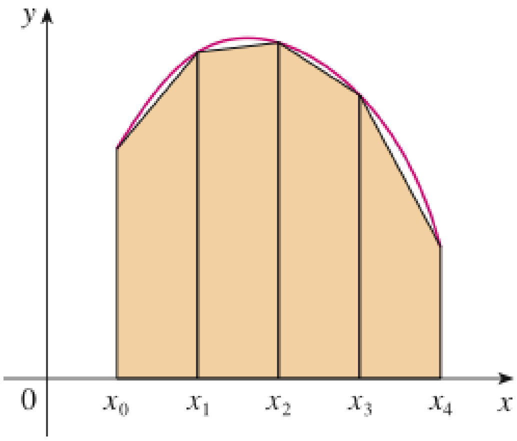
Trapezoid Rule.

The trapezoidal approximation to

$$\int_a^b f(x)dx \text{ with } \Delta x = \frac{b-a}{n}$$

is

$$T_n = \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)].$$



Example 3.17 Calculate an approximation to the integral

$$\int_0^3 x^2 dx$$

with $n = 6$ and $\Delta x = 0.5$ by using

(a) left-endpoint approximation
(b) right-endpoint approximation
(c) midpoint approximation
(d) trapezoidal approximation

Errors in Approximation:
The **error** E in using an approximation is defined to be the difference between the actual value and the approximation A. That is,

$$E = \int_a^b f(x)\,dx - A$$

It turns out that the size of the error depends on the second derivative of the function f, which measures how much the graph is curved.

The following fact is usually proved in a course on *numerical analysis* (MACM316), so we just state it here.

Theorem 3.6.1 — Error bounds. Suppose that $|f''(x)| \leq K$ for x in the interval $[a,b]$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

Note: $|f''(x)|$ "measures" how far the function is away from being a line; the length of the interval is part of the estimate; and, in both cases the error (bound) is quadratic in step-size $1/n$, i.e., if you double the number of points the error should be divided by approximately **4**.

Example 3.18 Since

$$\int_1^2 \frac{dx}{x} = \ln 2$$

the Trapezoidal and Midpoint Rules could be used to approximate $\ln 2$. Estimate the errors in the the Trapezoidal and Midpoint approximations of this integral by using $n = 10$ intervals.

Example 3.19

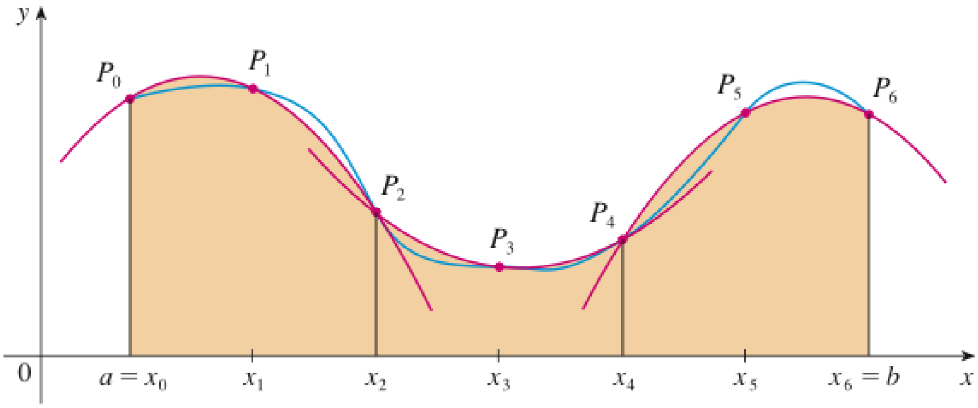
(a) Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{-x^2}\,dx$.

(b) Give an upper bound for the error involved in this approximation.

(c) How large do we have to choose n so that the approximation M_n to the integral in part (a) is accurate to within 0.00001?

Approximation using parabolic segments:

Let f be continuous on $[a,b]$ and divide the interval into an *even number n* subintervals of equal length $\Delta x = \frac{b-a}{n}$. Suppose the endpoints of these subintervals are, as usual, $a = x_0, x_1, x_2, \ldots, x_n = b$.



Let P_i be the point $(x_i, f(x_i))$. For each even number $i < n$ we approximate the area under the curve $y = f(x)$ over the interval $[x_i, x_{i+2}]$ by the area under the unique parabola that passes through the points P_i, P_{i+1}, and P_{i+2} over the same interval.

Simpson's Rule

Let f be continuous on $[a,b]$ and $\Delta x = \frac{b-a}{n}$ with n even.

Then we can approximate $\displaystyle\int_a^b f(x)\,dx$ by the sum

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

Example 3.20 Approximate $\displaystyle\int_0^3 \frac{dx}{1+x^4}$ by Simpson's Rule with $n = 6$.

Error in Simpson's Rule.
This time, the size of the error depends on the *fourth* derivative of f.

Theorem 3.6.2 — Error Bound in Simpson's Rule. Suppose that $|f^{(4)}(x)| \leq K$ for all x in the interval $[a,b]$. If E_S is the error in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

For Simpson, the size of the error depends on the *fourth* derivative of f; It measures how far away f is from being a cubic polynomial.

If you double n, the number of subintervals, the error will be divided by approximately **16** now!

Example 3.21 How large should we take n in order to guarantee that the Simpson's Rule approximation to $\int_1^2 \frac{1}{x}dx$ is accurate within 0.0001?

Additional Notes:

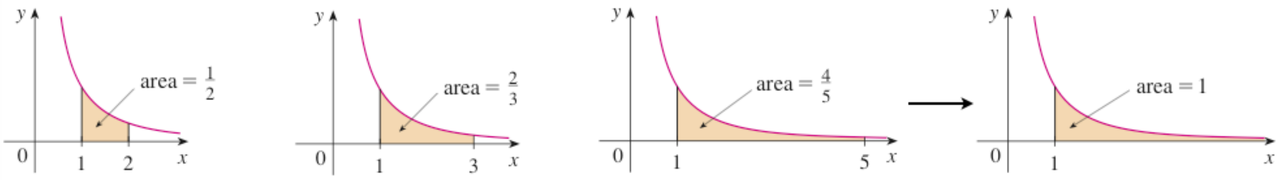
3.7 Improper Integrals

(This lecture corresponds to Section 7.8 of Stewart's *Calculus*.)

> *You and I are essentially infinite choice-makers. In every moment of our existence, we are in that field of all possibilities where we have access to an infinity of choices.*
>
> (Deepak Chopra, Indian ayurvedic Physician and Author, 1947-)

Motivating Problem Evaluate the area of the region bounded by the curves

$$y = \frac{1}{x^2}, \quad y = 0, \ x = 1.$$



Improper Integral of Type I:

(a) If $\int_a^t f(x)dx$ exists for all $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ provided that this limit exists (i.e. as a finite number).

(b) If $\int_t^b f(x)dx$ exists for all $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$ provided that this limit exists (i.e. as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_{-\infty}^a f(x)dx$ and $\int_a^\infty f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

Example 3.22 Investigate the improper integrals.

(a) $\displaystyle\int_{1}^{\infty} \frac{dx}{x}$

(b) $\displaystyle\int_{1}^{\infty} \frac{dx}{x^2}$

(c) $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt{1-x}}$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Motivating Problem Evaluate the area of the region bounded by the curves

$$y = \frac{1}{\sqrt{x}}, \quad y = 0, \quad x = 0, \quad x = 1.$$

Improper Integral of Type II

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided that this limit exists.

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided that this limit exists.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 3.23 Investigate the improper integrals.

(a) $\displaystyle\int_1^2 \frac{dx}{(x-2)^2}$

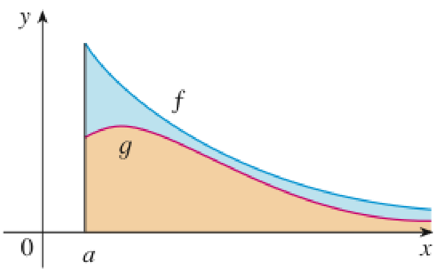
(b) $\displaystyle\int_0^2 \frac{dx}{(2x-1)^{2/3}}$

(c) $\displaystyle\int_0^1 \ln x \, dx$

Theorem 3.7.1 — Comparison Theorem. Suppose that f and g are continuous functions with $0 \le g(x) \le f(x)$ for $x \ge a$.

(a) If $\displaystyle\int_a^{\infty} f(x)dx$ is convergent then $\displaystyle\int_a^{\infty} g(x)dx$ is convergent.

(b) If $\displaystyle\int_a^{\infty} g(x)dx$ is divergent then $\displaystyle\int_a^{\infty} f(x)dx$ is divergent.



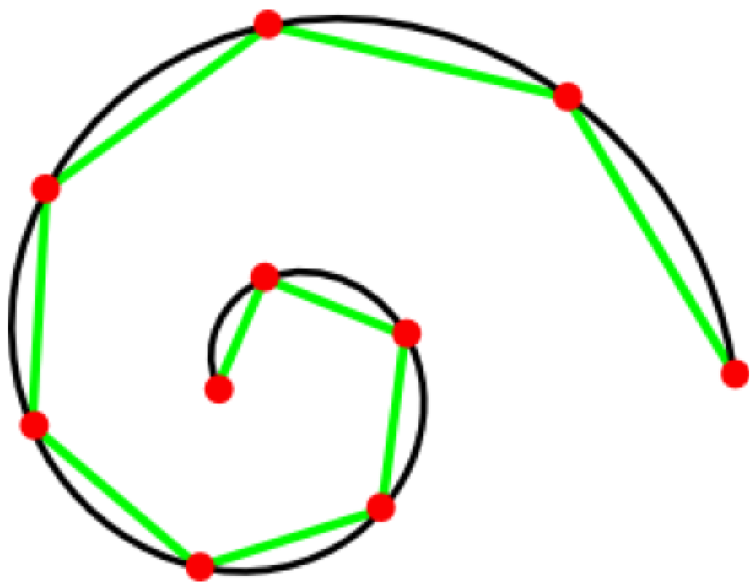
Example 3.24 Use the Comparison Theorem to determine if the following integrals are convergent or divergent.

(a) $\displaystyle\int_4^\infty \frac{dx}{\ln x - 1}$

(b) $\displaystyle\int_1^\infty e^{-x^2/2}dx$

Additional Notes:

4. Further Applications of Integration



In this chapter we continue looking at more applications of integration.

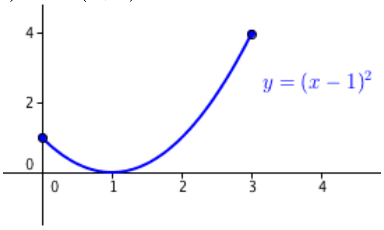
Topics we will cover are:
- arc length,
- surface area,
- integration along parametric curves.

4.1 Arc Length

(This lecture corresponds to Section 8.1 of Stewart's *Calculus*.)

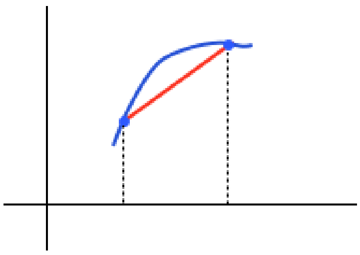
> *As usual, Ronaldinho takes the free kick. He sent the ball whistling into the air with his right foot. Just as ball looked to fly wide, it curled in a perfect arc and entered the net at the top right corner.*
>
> (From http://hvdofts.wordpress.com/2006/11/21/controversy-in-camp-nou/)

Motivating Problem Find the length of the arc of the parabola $y=(x-1)^2$ between the points $(0,1)$ and $(3,4)$.



Theorem 4.1.1 — The Arc Length Formula. If f' is continuous on $[a,b]$, then the length of the curve $y=f(x)$, $a\le x\le b$ is

$$L=\int_a^b \sqrt{1+[f'(x)]^2}dx.$$



Example 4.1 Find the length of the following arcs.

(a) $y = (x-1)^2$ between the points $(0,1)$ and $(3,4)$

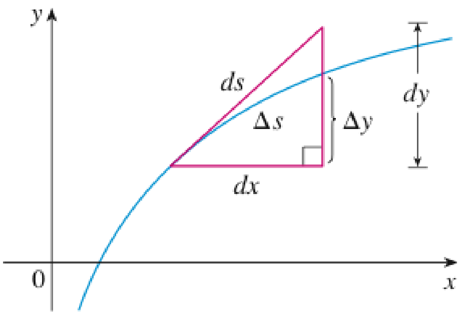
(b) $x = \frac{1}{6}y^3 + \frac{1}{2y}$, $1 \leq y \leq 2$

(c) $y = x^3$, $0 \leq x \leq 5$

The Arc Length Function. Let a smooth curve C have the equation $y = f(x)$, $a \leq x \leq b$. Let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$.

(a) Find the formula for $s(x)$.

(b) Find $\dfrac{ds}{dx}$, as well as the differential ds.



Example 4.2 Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

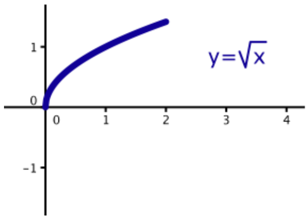
Additional Notes:

4.2 Area of a Surface of Revolution

(This lecture corresponds to Section 8.2 of Stewart's *Calculus*.)

Be like a duck. Calm on the surface, but always paddling like the dickens underneath.
(Michael Caine, British Actor, 1933-)

Motivating Problem Find the surface area of the paraboloid which is obtained by revolving the parabolic arc $y=\sqrt{x}$, $0 \le x \le 2$, about the x-axis.



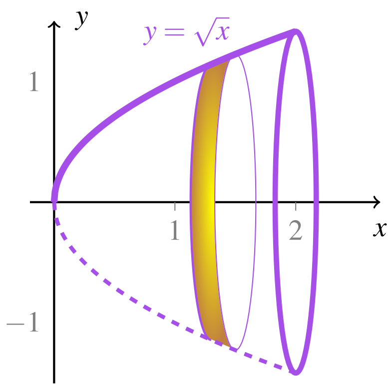
Surface Area
Let a smooth curve C be given by $y=f(x)$, $x \in [a,b]$.

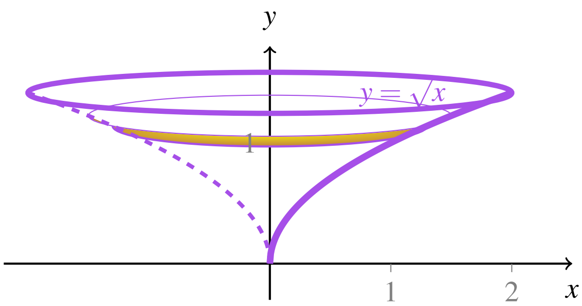
(a) The area of the surface obtained by rotating C about the x-axis is defined as

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'x)]^2}dx.$$

(b) The area of the surface obtained by rotating C about the y-axis is defined as

$$S=\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}dx.$$





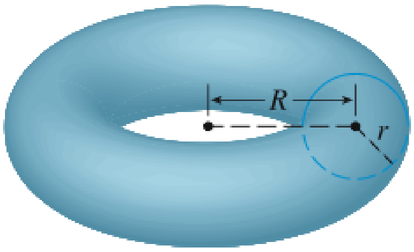
Area formulas for surfaces of revolution

Description of curve C	Revolution about x-axis	Revolution about y-axis
$y = f(x), x \in [a,b]$	$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$	$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx$
$x = g(y), y \in [c,d]$	$\int_c^d 2\pi y\sqrt{1+[g'(y)]^2}\,dy$	$\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$

Example 4.3 Find the area of the surface obtained by rotating the given arc about the corresponding axis.

(a) $y = \sqrt{x}$, $0 \le x \le 2$, about the x-axis
(b) $y = x^3$, $0 \le x \le 2$, about the x-axis
(c) $y = x^2$, $0 \le x \le \sqrt{2}$, about the y-axis

Example 4.4 Find the surface area of the torus.



Additional Notes:

4.3 Calculus with Parametric Curves

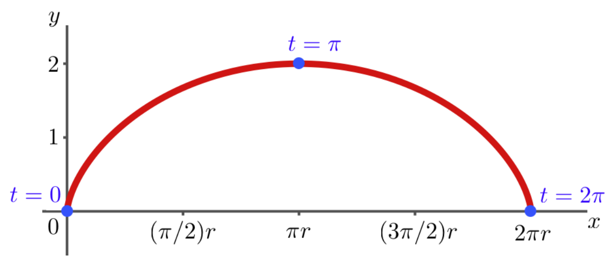
(This lecture corresponds to Section 10.2 of Stewart's *Calculus*.)

> *Contrary to common belief, the calculus is not the height of the so-called 'higher mathematics.' It is, in fact, only the beginning.*
>
> (Morris Kline, American mathematician, 1908-1992)

Motivating Problem Find the arc length of one arch of the cycloid

$$x = r(t - \sin t), y = r(1 - \cos t), 0 \leq t \leq 2\pi.$$

Also find the area under this arch.



Some previous results in a new context ...

Calculus with Parametric Curves:

Suppose the function $y(x)$, for $x \in [a,b]$, is defined by the parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t) \quad \text{for } t \in [\alpha, \beta]$$

and let C be the corresponding parametric curve.

(We assume that f and g satisfy all conditions that will guarantee that the function $y(x)$ has the necessary properties that allow for the existence of all listed integrals.)

1. If f and g are differentiable with $f'(t) \neq 0$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$.

2. If $y(x) \geq 0$, then the **area** under the curve C is given by

$$A = \int_a^b y(x)\, dx = \int_\alpha^\beta g(t) f'(t)\, dt$$

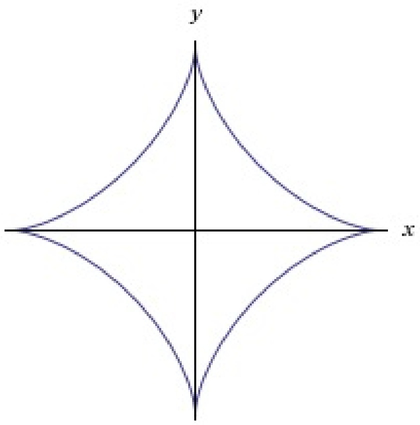
3. If f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the **length of the curve** C is given by

$$s = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

3. If $g(t) \geq 0$ then the **area of the surface** obtained by rotating C about the x-axis is given by

$$S = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_\alpha^\beta 2\pi g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

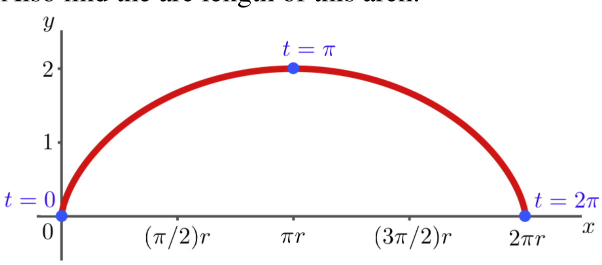
Example 4.5 Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ as a function of the parameter θ. At what points is the tangent horizontal. Vertical? At what points does that tangent have slope 1. What about slope -1?



Example 4.6 Find the area under one arch of the cycloid:

$$x = r(t - \sin t), y = r(1 - \cos t), 0 \le t \le 2\pi.$$

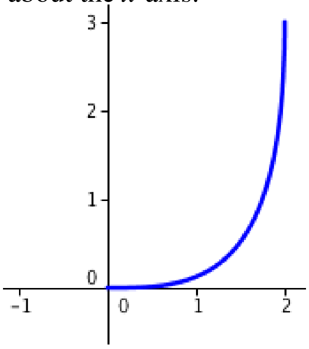
Also find the arc length of this arch.



Example 4.7 Find the area of the surface obtained by rotating the curve

$$x = 3t - t^3,\ y = 3t^2,\ \ 0 \le t \le 1$$

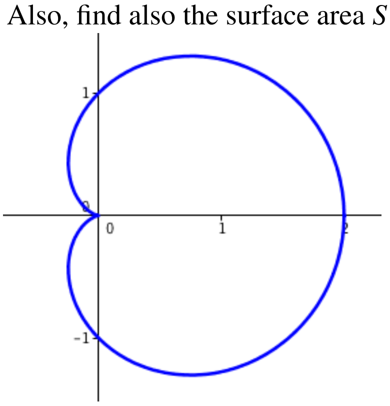
about the x-axis.



Example 4.8 Polar Coordinates are just parametric equations. Really!!
Find the arc length s of the cardioid with polar equation

$$r = 1 + \cos\theta.$$

Also, find also the surface area S generated by revolving the cardioid around the x-axis.

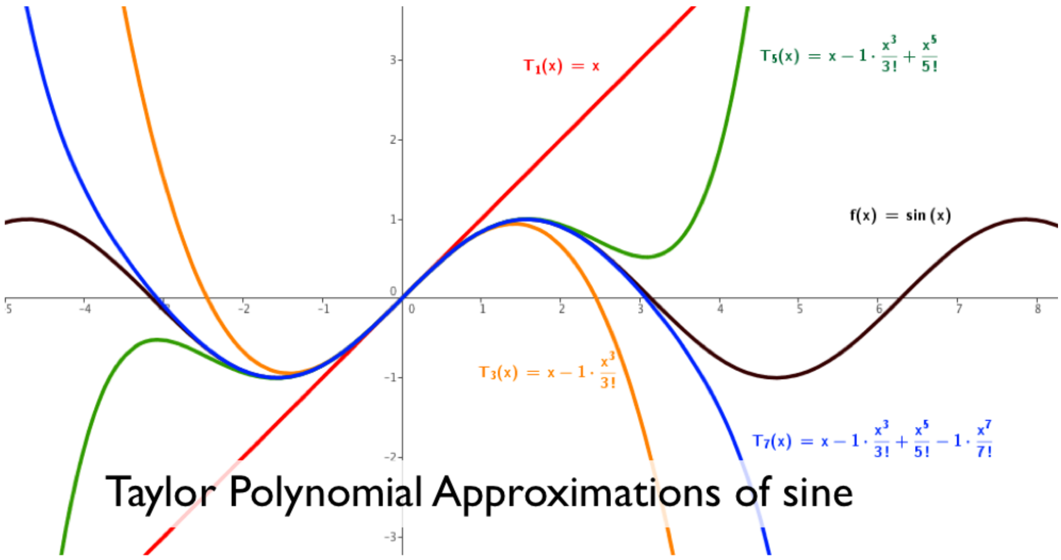


Additional Notes:

Part Three: Sequences and Series

5 Infinite Sequences and Series 125
5.1 Sequences
5.2 Series
5.3 The Integral Test and Estimates of Sums
5.4 The Comparison Test
5.5 Alternating Series
5.6 Absolute Convergence and the Ratio and Root Test
5.7 Strategy for Testing Series
5.8 Power Series
5.9 Representation of Functions as Power Series
5.10 Taylor and Maclaurin Series
5.11 Applications of Taylor Polynomials

5. Infinite Sequences and Series



In this chapter we consider infinite sums of numbers, and how to represent functions as infinite polynomials (power series).

Topics we will cover are:

- infinite sequences,
- infinite series,
- power series and Taylor series

5.1 Sequences

(This lecture corresponds to Section 11.1 of Stewart's *Calculus*.)

Mensa Puzzle. What number comes next in this sequence?

$$1 \quad 3 \quad 8 \quad 19 \quad 42 \quad ?$$

What is the 100th number in the sequence?

Definition 5.1.1 — Sequence. A **sequence** is a function whose domain is the set $\mathbb{Z}^+ = \{1,2,3,\ldots\}$ of positive integers.

If the function is $s : \mathbb{Z}^+ \to \mathbb{R}$, then the output $s(n)$ is usually written as s_n, we also write the whole sequence as $s = \{s_n\}$.

Note: Sometimes the domain of a sequence may be taken as $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$, in which case we write $\{s_n\}_{n=0}^{\infty}$.

Example 5.1

(a) Write out the first few terms of the sequence

$$\{\cos n\pi\}_{n=2}^{\infty}.$$

Is it possible to write this sequence in a different form?

(b) Graph the sequence $\left\{1 + \frac{(-1)^n}{n}\right\}$.

Definition 5.1.2 — Limit of a sequence. (Informal definition)
A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or it is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Definition 5.1.3 — Limit of a sequence. (Formal or rigorous definition: "ε-N definition")

A sequence $\{a_n\}$ has the **limit** L and we write

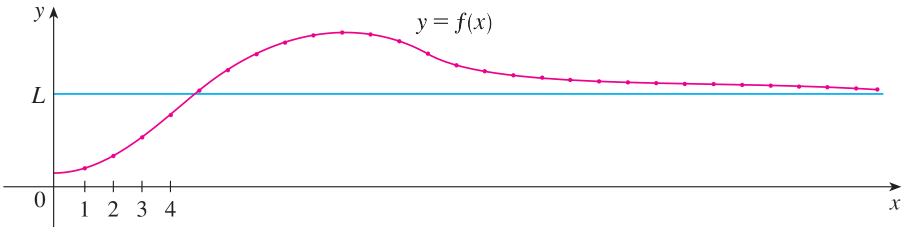
$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \text{ whenever } n > N.$$

Example 5.2 Is the sequence $\left\{\frac{2n}{n+3}\right\}$ convergent or divergent?

Theorem 5.1.1 Consider the sequence $f(n) = a_n$ where n is an integer.
If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.



Definition 5.1.4

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive number M there is an integer N such that

$$a_n > M \text{ whenever } n > N.$$

Theorem 5.1.2 — Facts about sequences.
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(b) $\lim\limits_{n\to\infty}(ca_n) = c\lim\limits_{n\to\infty} a_n$ (in particular, this means that $\lim\limits_{n\to\infty} c = c$)

(c) $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(d) $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, as long as $\lim\limits_{n\to\infty} b_n \neq 0$

(e) $\lim\limits_{n\to\infty}(a_n)^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ only for $p > 0$ and $a_n > 0$.

(f) If $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

(g) If $a_n \leq c_n \leq b_n$ for all $n \geq N$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$, then $\lim\limits_{n\to\infty} c_n = L$.

(h) If $\lim\limits_{n\to\infty} a_n = L$ and a function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$

Example 5.3

(a) Show that the sequence $\{\sqrt[n]{n}\}$ converges to 1.

(b) Is the sequence $a_n = \sin\left(\dfrac{n\pi}{2}\right)$ convergent or divergent?

Example 5.4

(a) Does the sequence $\left\{\frac{\cos(n\pi)}{n}\right\}$ converge or diverge?

(b) For what values of r is the sequence $\{r^n\}$ convergent?

Definition 5.1.5 A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \ldots$.
It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
It is called **monotonic** if it is either increasing or decreasing.

Example 5.5 Decide which of the following sequences is increasing, decreasing or neither.

(a) $a_n = 1 + \frac{1}{n}$

(b) $b_n = 1 - \frac{1}{n}$

(c) $c_n = 1 + \frac{(-1)^n}{n}$

Definition 5.1.6 A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

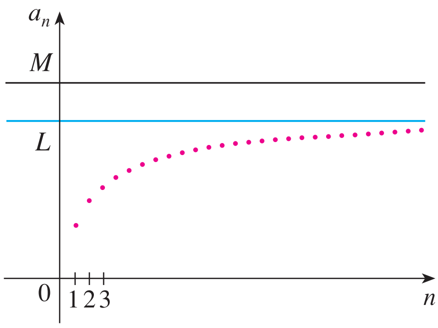
$$a_n \leq M \text{ for all } n \geq 1.$$

It is **bounded below** if there is a number m such that

$$m \leq a_n \text{ for all } n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Theorem 5.1.3 — Monotonic Sequence Theorem. Every bounded, monotonic sequence is convergent.



Example 5.6 Investigate the sequence $\{a_n\}$ that is defined recursively by

$$a_1 = \sqrt{6},\ a_{n+1} = \sqrt{6 + a_n}\ ,\quad \text{for } n \geq 1.$$

(example continued)

Additional Notes:

5.2 Series

(This lecture corresponds to Section 11.2 of Stewart's *Calculus*.)

> **Joke:** *An infinite crowd of mathematicians enters a bar. The first one orders a pint, the second one a half pint, the third one a quarter pint ... "I understand," says the bartender – and pours two pints.*

Definition 5.2.1 — Series. Suppose $\{a_n\}$ is a sequence of numbers. An expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

is called an **infinite series** and it is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n.$$

Definition 5.2.2 — Partial Sum. If $\sum_{i=1}^{\infty} a_i$ is a series then

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

is called its nth **partial sum**.

But does it make sense to "add infinitely many numbers"?
Not directly, so we imagine adding finitely many terms, but more and more terms each time, and look at what happens to these cumulative sums.

Definition 5.2.3 Given the series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots = s$$

The number s is called the **sum** of the series.

If the limit above does not exist, then the series is called **divergent**.

Example 5.7 The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ has partial sums $s_1 = 1$, $s_2 = 1.5$, $s_3 = 1.75$, $s_4 = 1.875\dots$ and in general it turns out that $s_n = 2 - \frac{1}{2^{n-1}}$.
Since $s_n \to 2$ as $n \to \infty$, the series is convergent and has sum 2.

Example 5.8 Show that the **geometric series** $\sum\limits_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \dots$ is convergent if $|r| < 1$ and its sum is

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r},\ |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.
(Here we are assuming $a \neq 0$, otherwise the series converges to 0 regardless of the value of r.)

Example 5.9 Determine whether the given series converges or diverges.

(a) $\sum_{n=1}^{\infty} \left(\frac{e}{10}\right)^n$

(b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3}{e}\right)^n$

Example 5.10 Express $0.5555\ldots$ as a rational number.

Example 5.11 Show that the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ is convergent and find its sum.

Example 5.12 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent.

Two Useful Results:

Theorem 5.2.1 — Test for Divergence.

(a) If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

(b) If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5.13 Show that

$$\sum_{n=1}^{\infty} n \sin(1/n)$$

is divergent.

Theorem 5.2.2 If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then $\sum ca_n$, $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ are also convergent, and

(a) $\sum ca_n = c\sum a_n$

(b) $\sum(a_n + b_n) = \sum a_n + \sum b_n$

(c) $\sum(a_n - b_n) = \sum a_n - \sum b_n$

Example 5.14 If $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2^n} - \frac{26}{(n+1)(n+2)}\right)$ is convergent, find its sum.

From Examples 5.2 and 5.2 , we know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$ are convergent, with sums 1 and $\frac{1}{2}$, respectively.
The given series is convergent, since it can be written as

$$5\sum_{n=1}^{\infty}\frac{1}{2^n} - 26\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = 5(1) - 26(\tfrac{1}{2}) = -8.$$

Additional Notes:

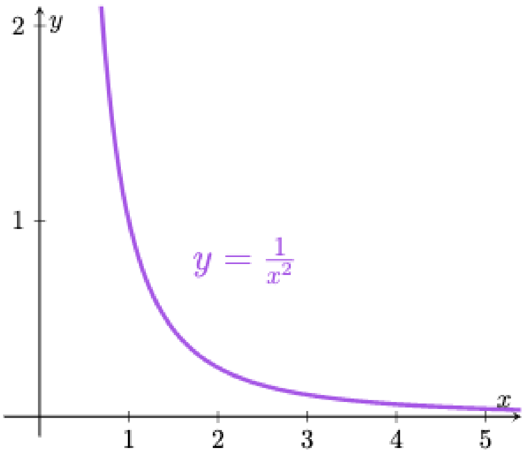
5.3 The Integral Test and Estimates of Sums

(This lecture corresponds to Section 11.3 of Stewart's *Calculus*.)

If you want to run, run a mile. If you want to experience a different life, run a marathon.
(From (Emil Zatopek, Czechoslovakian athlete,1922-2000)

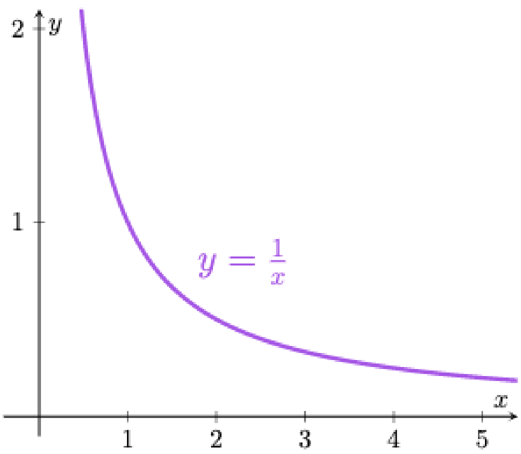
Motivating Problem Compare

$$\int_1^{\infty} \frac{dx}{x^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$



Motivating Problem Compare

$$\int_1^{\infty} \frac{dx}{x} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n}.$$



Theorem 5.3.1 — The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5.15 Is the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

convergent or divergent?

Example 5.16 Use the integral test to test the p-**series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ for convergence.

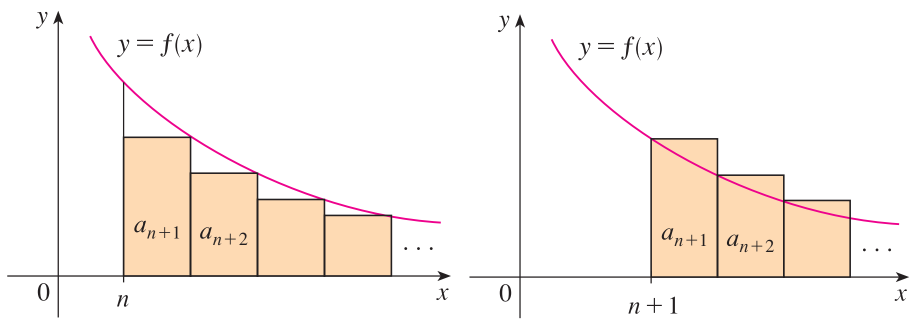
Remainder when using partial sums to estimate a series

If $\sum_{n=1}^{\infty} a_n = s$ is convergent then the n^{th} remainder is defined as

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$$

Theorem 5.3.2 — Remainder Estimate for the Integral Test. Suppose $f(k) = a_k$, where f is continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$



Example 5.17 In a previous example we showed that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converges. Determine how many terms you would need to add to find the value of this sum accurate to within 0.01. That is, how large must n be for the reminder to satisfy the inequality $R_n < 0.01$?

Additional Notes:

5.4 The Comparison Test

(This lecture corresponds to Section 11.4 of Stewart's *Calculus*.)

The will to win means nothing without the will to prepare.
(Juma Ikangaa, Tanzanian marathoner, 1957-)

Motivating Problem Test if
$$\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$$
is convergent.

Theorem 5.4.1 — The Comparison Test.
Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $0 \leq a_n \leq b_n$ for all n.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

Example 5.18 Test if
$$\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$$
is convergent.

Useful tip:
When applying the comparison test, you can often use geometric series or *p*-series.

Example 5.19 Test the series

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

for convergence.

Theorem 5.4.2 — The Limit Comparison Test.

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example 5.20 Test for convergence

(a) $\sum \frac{3n^2+n}{n^4+\sqrt{n}}$

(b) $\sum \frac{1}{2n+\ln n}$

Additional Notes:

5.5 Alternating Series

(This lecture corresponds to Section 11.5 of Stewart's *Calculus*.)

When you win, say nothing. When you lose, say less.
(Paul Brown, American football coach, 1908-1991)

Motivating Problem We have already seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. But what happens if we alternately add and subtract the terms instead?

Is the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

convergent or divergent?

Definition 5.5.1 — Alternating Series. If $\{b_n\}$ is a sequence of positive numbers then

$$b_1 - b_2 + b_3 - b_4 + \dots$$

is called an alternating series.

Theorem 5.5.1 — The Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots, \quad (b_n > 0)$$

satisfies

(a) $b_{n+1} \leq b_n$ for all n
(b) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Example 5.21 Test if

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

is convergent or divergent.

Example 5.22 Test for convergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+2n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{2} - \arctan n\right)$

Theorem 5.5.2 — Alternating Series Estimation Theorem. If $s = \sum_{n=1}^{\infty}(-1)^n b_n$ is the sum of an alternating series which satisfies

$$\text{(i)} \quad 0 \leq b_{n+1} \leq b_n \quad \text{and} \quad \text{(ii)} \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example 5.23 Use the fact that

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$$

to compute $e^{-1}$to four decimal places.

Additional Notes:

5.6 Absolute Convergence and the Ratio and Root Test

(This lecture corresponds to Section 11.6 of Stewart's *Calculus*.)

You cannot always run at your best.

(Bill Rodgers, American runner, 1947-)

Motivating Problem Test if

$$\sum_{n=1}^{\infty} \frac{n2^n}{n!}$$

is convergent or divergent.

Definition 5.6.1 A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Example 5.24 The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is *absolutely convergent* since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Definition 5.6.2 A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 5.25 — Alternating Harmonic Series. We have already seen that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent, but we know that $\sum \frac{1}{n}$ (the harmonic series) is divergent. Therefore we say the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *conditionally convergent*.

Theorem 5.6.1 — Absolute Convergence vs. Convergence. If a series $\sum a_n$ is absolutely convergent then it is convergent.

Example 5.26 Determine if the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2+2n+1}$$

is convergent or divergent.

Theorem 5.6.2 — Test for Absolute Convergence (Part 1).
The Ratio Test:

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example 5.27 Test for convergence, using the ratio test.

(a) $\sum_{n=1}^{\infty} \frac{n2^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem 5.6.3 — Test for Absolute Convergence (Part 2).

The Root Test:

(a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, the Root Test is inconclusive; no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$..

Example 5.28 Test for convergence, using the root test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Additional Notes:

5.7 Strategy for Testing Series

(This lecture corresponds to Section 11.7 of Stewart's *Calculus*.)

You have to be fast on your feet and adaptive or else a strategy is useless.
(Charles de Gaulle, French statesman, 1890-1970)

Given a series, the *Main Question* is: convergent or divergent?

To help solve this question, we have the following tests:

(a) Test for Divergence (p. 136): if $a_n \not\to 0$ as $n \to \infty$, then series diverges

(b) If $a_n \geq 0$, then we could use these tests:
- geometric series (p. 134) or p-series (p. 140)
- telescoping series (p. 135 for an example)
- integral test (p. 140)
- comparison test (p. 143)
- limit comparison test (p. 145)

(c) If a_n is alternating in sign, try the Alternating Series Test (p. 147)

(d) If a_n is any real number, then:
- check absolute convergence (p. 151)
- try the Ratio Test (p. 153)
- try the Root Test (p. 154)

Time Machine. Test for convergence.

(a) **Previous final exam 1**

(i.) $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$

(ii.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+\sin n}}$

(iii.) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln\sqrt{n}}$

(b) **Previous final exam 2**

(i.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

(ii.) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(iii.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$

(iv.) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

(c) **Previous final exam 3**

(i.) $\sum_{n=1}^{\infty}(\arctan(n+1)-\arctan n)$

(ii.) $\sum_{n=1}^{\infty}\left(\frac{-3}{\pi}\right)^n$

(iii.) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

(d) **Previous final exam 4**

(i.) $\sum_{n=1}^{\infty}\frac{1}{n^p}$

(ii.) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$

(iii.) $\sum_{n=1}^{\infty}\frac{1}{(2n^2+1)^{2/3}}$

(iv.) $\sum_{n=1}^{\infty}\frac{2^n}{3^n-n}$

(e) **Previous final exam 5**

(i.) $\sum_{n=1}^{\infty}\frac{4^n}{3^{2n-1}}$

(ii.) $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{1/n}}$

(iii.) $\sum_{n=1}^{\infty}\frac{2^n}{(2n+1)!}$

(iv.) $\sum_{n=1}^{\infty}\frac{\tan^{-1}n}{n\sqrt{n}}$

(v.) $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^n$

(f) **Previous final exam 6**

(i.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(ii.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^5+4}}$

(iii.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2+1}$

(iv.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)5^n}{n3^{2n}}$

(g) **Previous final exam 7**

(i.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5}$

(ii.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(h) **Previous final exam 8**

(i.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(1+n^2)^3}$

(ii.) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(iii.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)!}{2^{2n-1}}$

Additional Notes:

5.8 Power Series

(This lecture corresponds to Section 8.1 of Stewart's *Calculus*.)

Knowledge is power.

(Francis Bacon, English Philosopher, 1561-1626)

Definition 5.8.1 — Power Series. Recall that polynomial is a function of the form

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

where n is a nonnegative integer, and the numbers $c_0, c_1, \ldots, c_n$ are constants called coefficients of the polynomials.

Similarly, we define a **power series** to be a function of the form

$$\sum_{n=0}^{\infty} c_nx^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

where x is a variable and the numbers $c_0, c_1, \ldots$ are constants called the **coefficients** of the series. A **power series in $\mathbf{x - a}$**, or a **power series centered at a**, has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots.$$

Example 5.29 For what values of $x \in \mathbb{R}$ is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-2)^n}{n \cdot 4^n}$$

convergent?

Example 5.30 The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the *Bessel function of order 1*. What is the domain of J_1.
(Note that 0! is by definition equal to 1.)

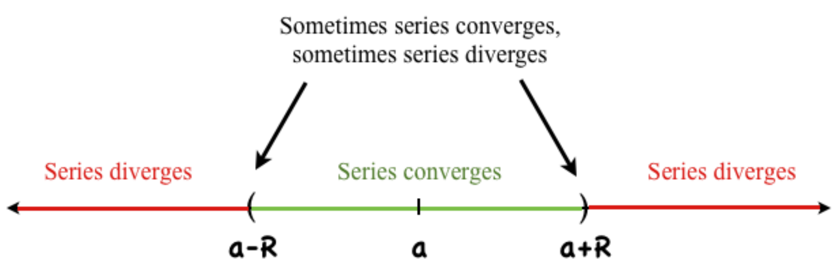
Where does a power series converge?

Theorem 5.8.1 For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(a) The series converges only when $x = a$.
(b) The series converges for all $x \in \mathbb{R}$.
(c) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Terminology.

- R - the **radius of convergence**
- the **interval of convergence** - the interval that consists of all values of x for which the series converges



Example 5.31 Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{(2n)!}$

Additional Notes:

5.9 Representation of Functions as Power Series

(This lecture corresponds to Section 11.9 of Stewart's *Calculus*.)

I don't want to imitate life in movies; I want to represent it.
(Petro Almodóvar, Spanish film maker, 1949-)

Motivating Problem Can the function $f(x) = \dfrac{1}{1+x}$ be written as a power series?

Definition 5.9.1 — Representation as a series. Let I be the interval of convergence for the power series $\sum_{n=0}^{\infty} c_n x^n$. For each $x \in I$, let $f(x)$ denote the series; that is,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \qquad \text{if } x \in I$$

Then we call $\sum_{n=0}^{\infty} c_n x^n$ a **power series representation** of $f(x)$.

Example 5.32 Find a power series representation of the following functions.

(a) $f(x) = \dfrac{1}{1+4x^2}$

(b) $g(x) = \dfrac{x}{9-x^2}$

Theorem 5.9.1 — Term-by-term differentiation or integration.

Suppose the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$. Then, the function f defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

(a) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(b) $\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Both these series have radii of convergence equal to R.

Example 5.33 Find a power series representation of the following functions.

(a) $f(x) = \frac{1}{(1-x)^2}$

(b) $g(x) = \ln(1+x)$

(c) $h(x) = \arctan x$

Additional Notes:

5.10 Taylor and Maclaurin Series

(This lecture corresponds to Section 11.10 of Stewart's *Calculus*.)

I easily judged that the book of Taylor would please you very little. It seems to me that such a writer is not at all fit to carry out the office of Secretary of the Royal Society.
(Gottfried Wilhelm Leibniz (also Leibnitz or von Leibniz), German mathematician, 1646-1716)

Motivating Problem Suppose the function f has a power series representation with radius of convergence R, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for all } x \text{ such that } |x-a| < R,$$

Can we express the coefficients c_n in terms of the function f?
(Hint: what is the n^{th} derivative of f, evaluated at $x = a$? That is, calculate $f^{(n)}(a)$.)

Theorem 5.10.1 — Power series representation is unique. If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for all } x \text{ such that } |x-a| < R,$$

then its coefficients are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Here we adopt the convention that $0! = 1$ and $f^{(0)}(x) = f(x)$.

So *if* a function f has a power series representation at a, then this representation *must be*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \tfrac{f'(a)}{1!}(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \cdots$$

and this representation is called the **Taylor series of the function f at a**.
For the special case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) \ + \ \tfrac{f'(0)}{1!}x + \tfrac{f''(0)}{2!}x^2 + \cdots$$

and this is called the **Maclaurin series** of $f(x)$.

Example 5.34 Find the Maclaurin series of the following functions.
(a) $f(x) = e^x$
(b) $f(x) = \cos x$

Some Terminology

(a) $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is the nth-degree **Taylor polynomial** of f at a

That is, $T_n(x) \;=\; f(a) \;+\; \frac{f'(a)}{1!}(x-a) \;+\; \cdots \;+\; \frac{f^{(n)}(a)}{n!}(x-a)^n$

Notice that $\lim_{n\to\infty} T_n(x) \;=\; \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$, the Taylor series of f.

(b) The **remainder** of the Taylor series is defined as $R_n = f(x) - T_n(x)$.

Theorem 5.10.2 Suppose $f(x) = T_n(x) + R_n(x)$, where T_n and R_n are as above. If

$$\lim_{n\to\infty} R_n(x) = 0, \quad \text{for } |x-a| < R,$$

then f is equal to the sum of its Taylor series on the interval $(a-R, a+R)$.

Bounds on the size of the remainder.
To show that any specific function f does have a power series representation, we must prove that $\lim_{n\to\infty} R_n(x) = 0$.

To do this, we usually use the following two facts.

Fact 1: Taylor's Inequality

If

$$\left| f^{(n+1)}(x) \right| \leq M \text{ for } |x-a| \leq d$$

then the remainder of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \leq d.$$

Fact 2. For every real number x, we have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

Example 5.35 Prove

(a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for every real number x

(b) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, for every real number x

Theorem 5.10.3 — Some important power series representations.
These Maclaurin series can be derived just as in the previous examples.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1+x+x^2+x^3+\ldots && (-1,1)\\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots && (-\infty,\infty)\\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} &&= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots && (-\infty,\infty)\\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} &&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots && (-\infty,\infty)\\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} &&= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots && (-1,1)\\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} &&= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots && (-1,1]
\end{aligned}$$

Example 5.36 Find the Maclaurin series for the following functions.

(a) $f(x) = x^2 e^{-3x}$
(b) $g(x) = \sin(x^2)$
(c) $h(x) = \dfrac{x}{9-x^2}$

Example 5.37 Find the sum of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots.$

Example 5.38 Use series to evaluate

$$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}.$$

Example 5.39 Find the Taylor series for the following functions centered at the given value of a.

(a) $f(x) = e^{-x}$, $a = 1$
(b) $g(x) = \sin(2x)$, $a = \pi$

Additional Notes:

5.11 Applications of Taylor Polynomials

(This lecture corresponds to Section 11.11 of Stewart's *Calculus*.)

Even if I don't finish, we need others to continue. It's got to keep going without me.
(Terry Fox, Canadian hero, 1958-1981)

Reminder If f has a power series representation at a then

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

and this representation is called the **Taylor series of the function f at a**.

Reminder — Taylor's Inequality. If

$$\left|f^{(n+1)}(x)\right|\leq M \text{ for } |x-a|\leq d$$

then the remainder of the Taylor series satisfies the inequality

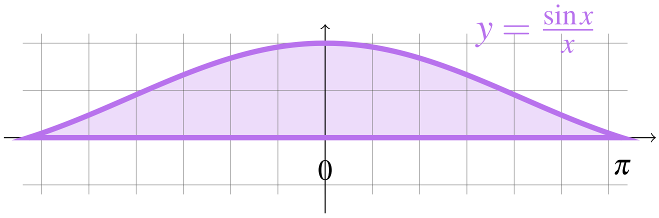
$$|R_n(x)|\leq\frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a|\leq d.$$

Example 5.40

(a) Approximate $f(x)=x^{2/3}$ by a Taylor polynomial with degree 3 at the number $a=1$.

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x)\approx T_3(x)$ when $0.8\leq x\leq 1.2$.

Example 5.41 Approximate the area between the curve $y = \dfrac{\sin x}{x}$ and the x-axis for $-\pi \le x \le \pi$.



Additional Notes:

IV

Part Four: Differential Equations

6 A First Look at Differential Equations . . 181

6.1 Modeling with Differential Equations, Directions Fields

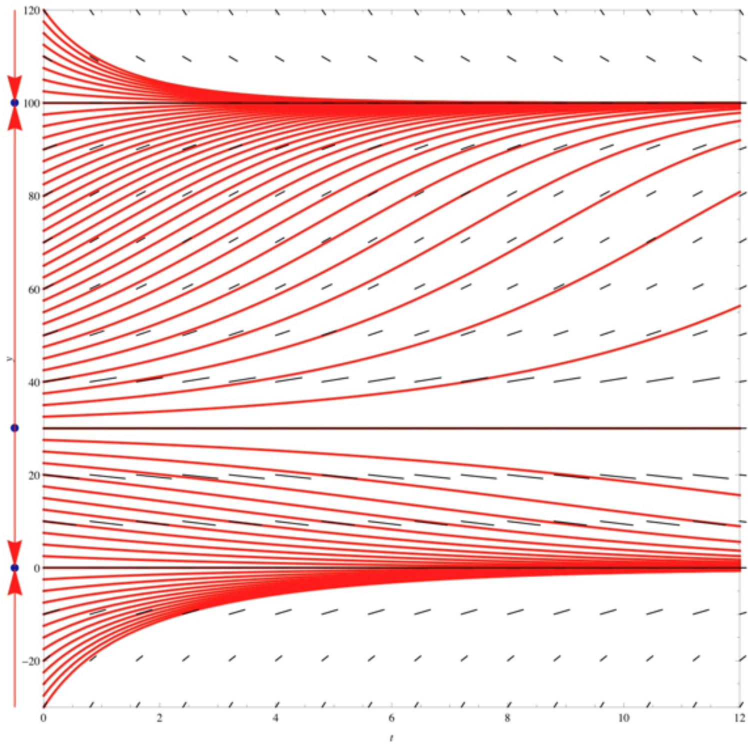
6.2 Separable Equations

6.3 Models for Population Growth

Natural Growth Model - revisited

Logistic Growth Model

6. A First Look at Differential Equations



In this chapter we consider how to use integration to find solutions to differential equations.

Topics we will cover are:

- examples of differential equations (DEs),
- visualizing a DE and solution using direction fields,
- techniques for finding the solution to a DE.

6.1 Modeling with Differential Equations, Directions Fields

(This lecture corresponds to Section 9.1 and the *Direction Fields* part of 9.2 of Stewart's *Calculus*.)

> *Once you learn the concept of a differential equation, you see differential equations all over, no matter what you do...If you want to apply mathematics, you have to live the life of differential equations. When you live this life, you can then go back to molecular biology with a new set of eyes that will see things you could not otherwise see.*
>
> (Gian-Carlo Rota in 'A Mathematician's Gossip', Indiscrete Thoughts (2008), 213)

Motivating Problem — Velocity of a Falling Object (considering air friction).
Imagine a sky-diver in free fall after jumping out of a plane. There are two forces acting on the falling mass: gravity ($F_g = mg$) and air friction ($F_a = -\gamma v$). Assume air friction is proportional to the velocity of the object.

The physical law that governs the motion of objects is Newton's second law, which states $F = ma$, where m is the mass of the object, a its acceleration, and F the net force on the object (in our case this is $F_g - F_a$).
Show that the velocity of the sky-diver satisfies this *differential equation*

$$m\frac{dv}{dt} = mg - \gamma v$$

where g and γ are constants.

Examples of some differential equations:

Type	Form
antidifferentiation	$\frac{dy}{dx} = x + \sin x$
natural growth	$\frac{dP}{dx} = kP$
Newton's Law of Cooling/Heating	$\frac{dT}{dt} = k(T - M)$
logistic growth	$\frac{dT}{dt} = P(1 - P)$
object falling under force of gravity (ignore friction)	$\frac{d^2s}{dt^2} = -g$

Terminology for Differential Equations:

- A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.
- The **order** of a differential equation is the order of the highest derivative that occurs in the equation.
- A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.
- To **solve** a differential equation means to find all possible functions that satisfy the equation.
- An **initial value problem** (IVP) is a differential equation together with an **initial condition**, which is just some specified value that the function must satisfy. An initial condition is presented in the form $y(t_0) = y_0$, which says we want the function y which satisfies the differential equation and has value y_0 at $t = t_0$.

Example 6.1 Show that the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$ has solutions $y = \sqrt{x^2+c}$.

Example 6.2

(a) Show that $y = e^{2t}$ is a solution to the second-order differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0.$$

(b) Show that $y = e^{-3t}$ is another solution.

Example 6.3 Find the value of c so that $y = \sqrt{x^2+c}$ is a solution to the *initial value problem*

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = 3$$

Direction Fields (also known as Slope Fields)

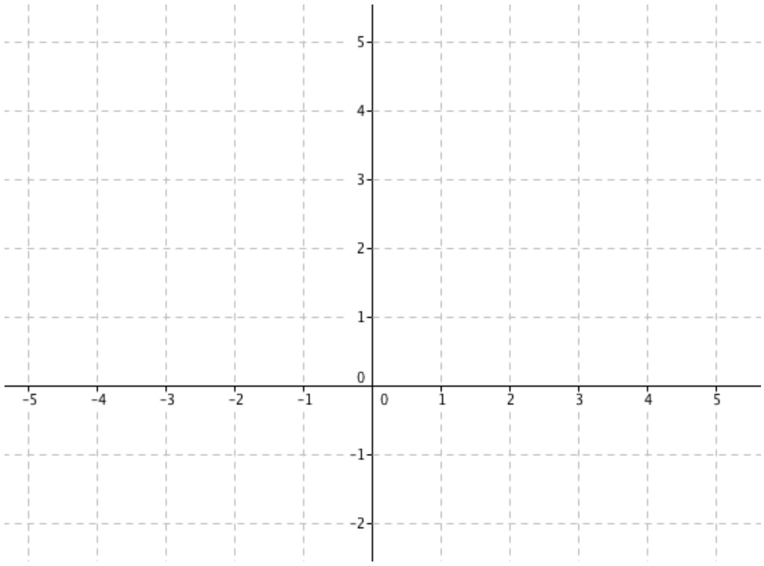
We now look at a visual approach for first-order differential equations.

Consider the differential equation

$$\frac{dy}{dt} = 3 - y.$$

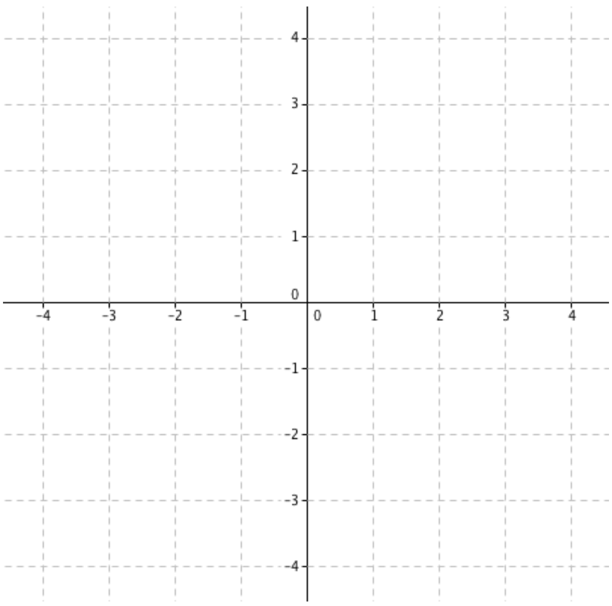
If $y = f(t)$ is a solution to this differential equation fill out values in the following table. Use this information to sketch a graph of f.

t	y	$\frac{dy}{dt} = f'(t)$
-3	1	
-3	2	



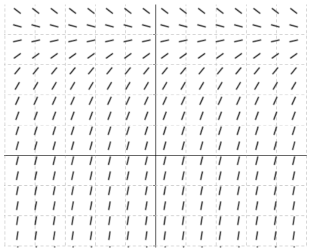
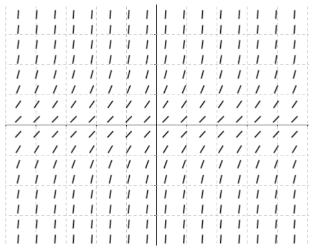
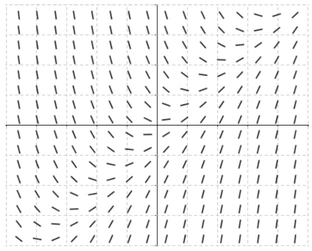
Example 6.4 Using the direction field, guess the form of the solution curves of the differential equation

$$\frac{dy}{dx} = \frac{-x}{y}.$$



Example 6.5 Match the differential equation with its corresponding slope field.

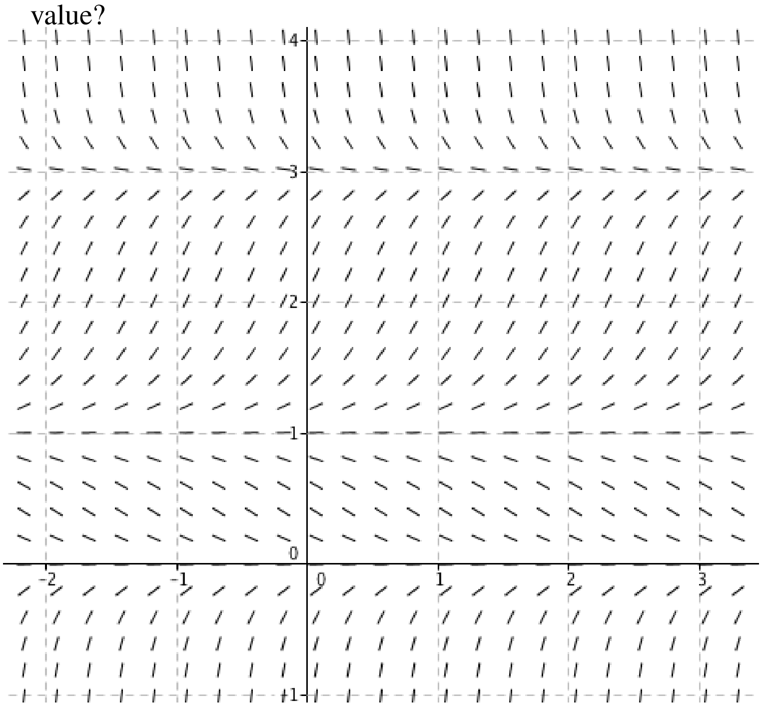
(a) $y' = 1 + y^2$ (b) $y' = x - y$ (c) $y' = 4 - y$

(i)  (ii)  (iii) 

Example 6.6 The slope field for $\dfrac{dy}{dt} = -y(y-1)(y-3)$ is given below.

(a) Sketch the solution curves satisfying the initial conditions
  (a) $y = 2$ when $t = 0$
  (b) $y = 0.5$ when $t = 0$

(b) What is the long-run behaviour of y? For example does $\lim_{t\to\infty} y$ exists? If so, what is its value?



Additional Notes:

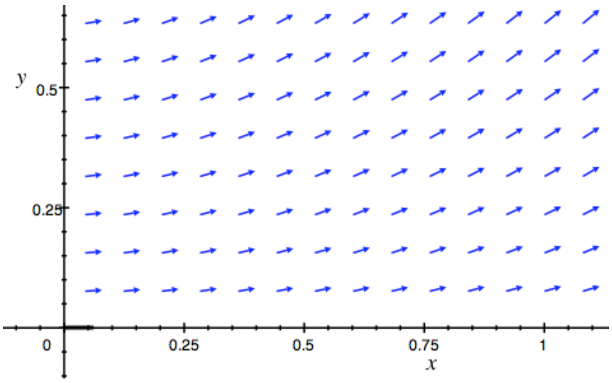
6.2 Separable Equations

(This lecture corresponds to Section 9.3 of Stewart's *Calculus*.)

> *Ideologies separate us. Dreams and anguish bring us together.*
> (Eugene Ionesco, Romanian born French dramatist, 1909-1994)

Motivating Problem Solve the differential equation

$$\frac{dy}{dx} = \sqrt{xy},\ x > 0,\ y > 0.$$

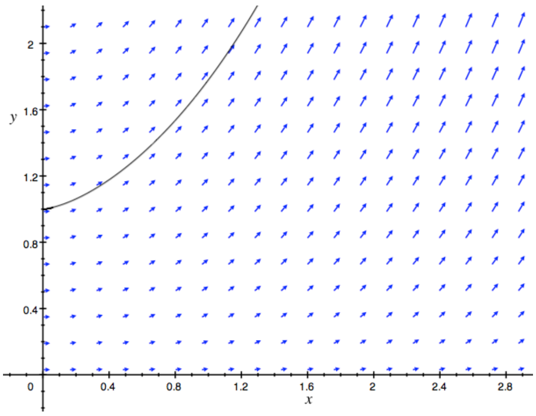


Definition 6.2.1 — Separable Equation. A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a product of a function of x and a function of y:

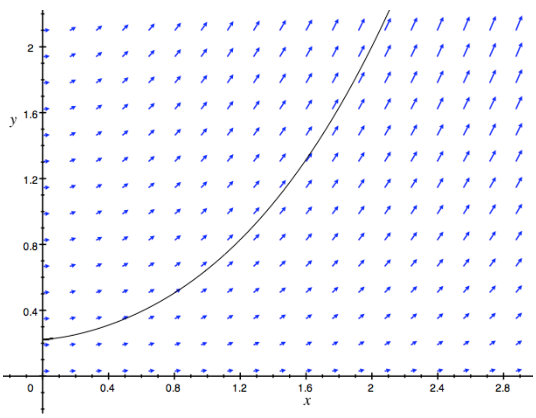
$$\frac{dy}{dx} = f(x) \cdot g(y).$$

Example 6.7 Solve the initial value problems:

(a) $\dfrac{dy}{dx} = \sqrt{xy}$, $y(0) = 1$;

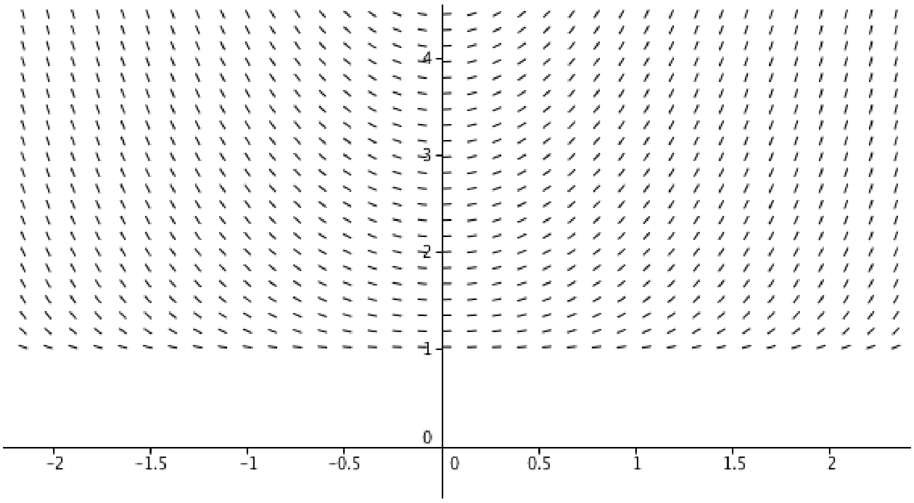


(b) $\dfrac{dy}{dx} = \sqrt{xy}$, $y(2) = 2$;

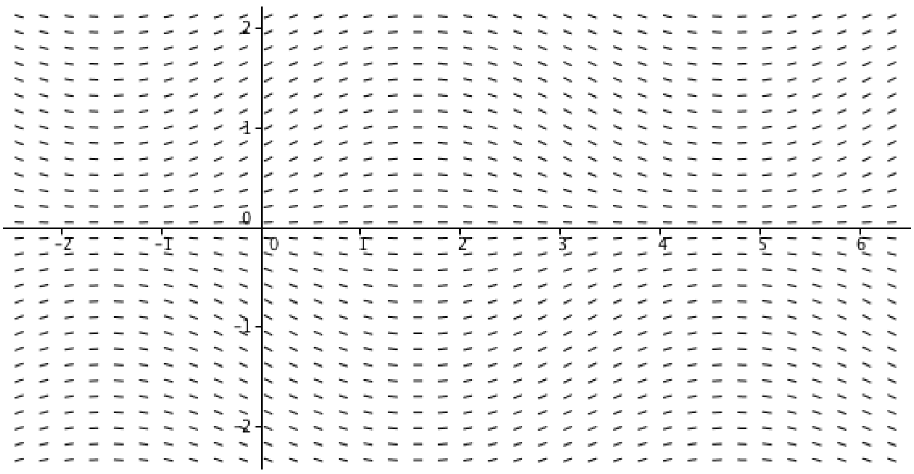


Example 6.8 Find general solutions.

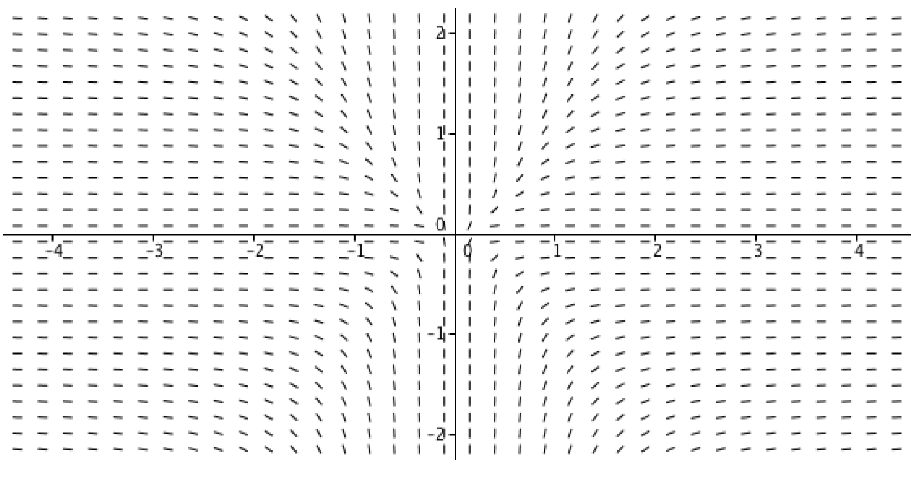
(a) $\dfrac{dy}{dx} = 2x\sqrt{y-1}$



(b) $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$



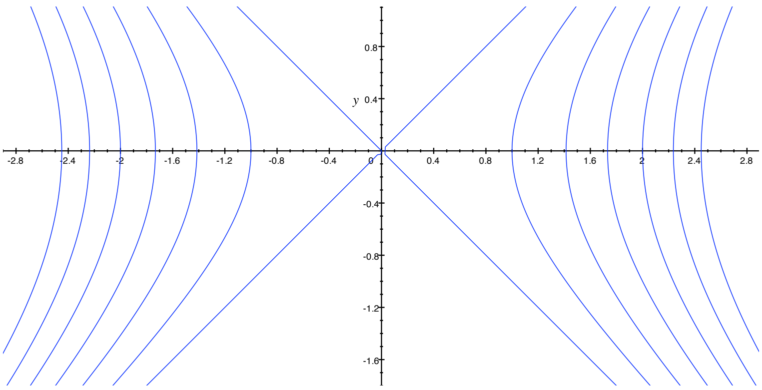
Example 6.9 Find an equation of the curve that passes through the point $(1,1)$ and whose slope at (x,y) is y^2/x^3.



Definition 6.2.2 — Orthogonal Trajectories.
An **orthogonal trajectory** of a given family of curves is a curve that intersects each member of the given at right angles.

Example 6.10 Find the orthogonal trajectories of the family of the curves

$$x^2 - y^2 = k.$$



Example 6.11 — A Mixing Problem.
A tank contains 1000L of pure water. Brine that contains 0.05 kg of salt per litre of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per litre of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

Additional Notes:

6.3 Models for Population Growth

(This lecture corresponds to Section 9.4 of Stewart's *Calculus*.)

> *A finite world can support only a finite population; therefore, population growth must eventually equal zero. Of course, a positive growth rate might be taken as evidence that a population is below its optimum.*
>
> (Garrett James Hardin, 1915 - 2003. American ecologist)

6.3.1 Natural Growth Model - revisited

The **Natural Growth Model** for population growth assumes that the population P at time t changes at a rate proportional to its size at any given time t. This can be written as

$$\frac{dP}{dt} = kP \qquad \text{Natural Growth Model}$$

where k is a constant.

6.3.2 Logistic Growth Model

The Natural Growth Model implies that the population would grow exponentially indefinitely. However the model must break down at some point since the population would eventually outstrip the food supply. In searching for an improvement we should look for a model whose solution is approximately an exponential function for small values of the population, but which levels off later.

The **Logistic Growth Model** for a population $P(t)$ at time t is based on the following assumptions:

- The growth rate is initially close to being proportional to size:

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

- The environment is only capable of a maximum population in the long run, this is called the **carrying capacity**, usually denoted by M:

$$\lim_{t\to\infty} P(t) = M.$$

The simplest expression for the growth rate that incorporates these assumptions is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad \text{Logistic Growth Model}$$

where k is a constant.

Additional Notes:

V Exam Preparation

7 Review Materials for Exam Preparation 199
7.1 Midterm 1 Review Package
7.2 Midterm 2 Review Package
7.3 Final Exam Practice Questions

7. Review Materials for Exam Preparation

Students frequently ask for previous semesters exams to help prepare for midterms and the final. That is what this section is for - it contains questions from previous exams. Think of this as a slightly longer "practice exam". The most efficient way to use this section is to find a quiet place to sit, free of distractions and reference material - in other words, put yourself in environment that simulates the actual exam. Give yourself 50 minutes for a midterm, 3 hours for a final, and write out your full solutions in that time period. Sticking precisely to the time limit isn't essential, these are designed to be longer than the actual exam. So if you can do them in 50mins/3hrs, or even slightly over that, you will have no issues with the time limit of the actual exam. Once you've completed the exam, compare your solutions with the answers provided. If you have troubles with any of the questions, seek help in the Calculus Workshop.

To prepare for an exam you should: (i) read all sections of the textbook again; (ii) go through all the homework questions again and make sure you can do every single one of them, (iii) work through more problems from the textbook. Only once you have done all this should you attempt the following questions for the specific exam.

7.1 Midterm 1 Review Package

Two weeks before the date of the exam an announcement will be posted on Canvas detailing which sections will be covered on the midterm.

Make sure you know the **definitions** of the terms: Riemann sum, definite integral, indefinite integral, even function, odd function, substitution rule, and the statements of the **theorems**: Fundamental Theorem of Calculus, Net Change Theorem. Also, you should know all the properties of integrals in Lecture 1.2, and the table of indefinite integrals in Lecture 1.4.

Make sure you review ALL the questions from the first 3 homework assignments. It is expected that you will know how to do all of these at the time of the midterm.

1. Compute the following integrals.
   (a) $\displaystyle\int_0^{\sqrt{\pi/3}} x\sin\left(x^2\right)\,dx$

   (b) $\displaystyle\int e^{\cos t}\sin t\,dt$

(c) $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\, dx$

(d) $\displaystyle\int_0^{e^{\pi/2}} f(x)\, dx$ where $f(x) = \begin{cases} e^{2x} & \text{if } 0 \le x \le 1 \\ \dfrac{\cos(\ln x)}{x} & \text{if } 1 < x \le e^{\pi/2} \end{cases}$

2. **True or False.** Justify your answers.
   (a) If f and g are continuous on $[a,b]$ then

$$\int_a^b f(x)g(x)\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right)$$

.

   (b) All continuous functions have antiderivatives.

(c) If f is continuous on $[a,b]$, then $\dfrac{d}{dx}\left(\displaystyle\int_a^b f(x)\,dx\right) = f(x)$.

(d) $\displaystyle\int_1^3 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^3\frac{2}{n}$

(e) If $\displaystyle\int_{-\pi}^{10} f(x)\,dx = 5$ then $\displaystyle\int_{10}^{-\pi} f(x)\,dx = -5$.

(f) $\displaystyle\int_{-a}^{a} Ax^2+Bx+C\,dx = 2\int_0^a Ax^2+C\,dx$, where A, B, C are constants.

3. Find the derivative of the function $g(x) = \displaystyle\int_{x^2}^{e^x}\frac{\ln t}{2t+1}\,dt$.

4. Compute the definite integral $\int_1^3 \left(x^2 - 1\right) dx$ by **using the definition of the definite integral** as a limit of a right-hand Riemann sum.
(Begin by finding explicit expressions for Δx and x_i in terms of n and i.)

5. Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$ about the line $y = -1$.
   To do this, use the *washer* method and sketch a typical washer.

6. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

   (Begin by sketching the region by hand, and determining the points of intersection of the two curves.)

Answers:

Only answers are provided here. You are expected to provide fully worked out solutions. If you need help with solving any of these problems please visit the Calculus Workshop or the Discussion board forum in Canvas.

1. (a) $1/4$ (b) $-e^{\cos t}$ (c) $-\frac{2}{15}(8+4x+3x^2)\sqrt{1-x}$ (d) $\frac{1}{2}(e^2+1)$
2. (a) F (b) T (c) F (d) T (e) T (f) T
3. $g'(x) = \frac{xe^x}{2e^x+1} - \frac{4x\ln x}{2x^2+1}$
4. $20/3$
5. $\frac{257}{210}\pi$
6. $2-\frac{\pi}{4}$

7.2 Midterm 2 Review Package

Two weeks before the date of the exam an announcement will be posted on Canvas detailing which sections will be covered on the midterm.

It is expected that you will know:

- the antiderivatives in the table of Lecture 3.5 (page 89)
- the trigonometric identities:

$$\cos^2\theta+\sin^2\theta=1 \qquad 1+\tan^2\theta=\sec^2\theta$$
$$\sin(2\theta)=2\sin\theta\cos\theta \qquad \cos(2\theta)=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$$

- the midpoint, trapezoid and Simpson's Rule and how to apply them. You will not be required to memorize the error bounds, but you should know how to use/apply the error bounds.

Make sure you review ALL the questions from the homework assignments. It is expected that you will know how to do all of these at the time of the midterm.

1. Compute the following integrals.

   (a) $\displaystyle\int_0^{\pi/3} x\sin x\,dx$

   (b) $\displaystyle\int x^5 e^{-x^3}\,dx$

(c) $\displaystyle\int \frac{\cos x}{\sin^2 x - \sin x}\, dx$

(d) $\displaystyle\int \frac{\ln(\tan x)}{\sin x \cos x}\, dx$

2. **True or False.** Justify your answers.

   (a) The antiderivative of $\dfrac{x^2+x+1}{x^3+x}$ involves an arctan term.

   (b) The integral $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ converges.

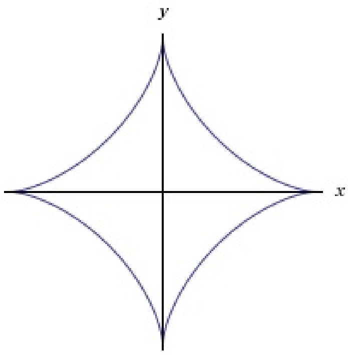
   (c) The arc length differential ds is given by $ds = \sqrt{1+(f'(x))^2}\,dx$ for $y = f(x)$.

   (d) $\displaystyle\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx$ represents the area of the surface obtained by rotating $y = f(x)$, $a \le x \le b$ about the y-axis.

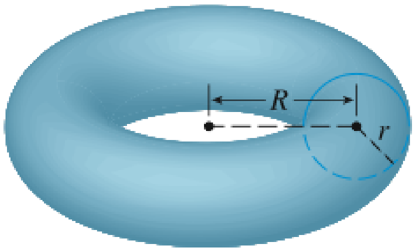
   (e) If $\displaystyle\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

3. Show that the circumference of a circle of radius r is $2\pi r$.

4. Determine the area enclosed by the astroid $x = a\cos^3\theta, y = a\sin^3\theta$, and the length of its perimeter.



5. Consider a **torus** with inner radius r and outer radius R as shown in the diagram.
   (a) Show the volume is $2\pi^2 r^2 R$ using the **washer method**.
   (b) Show the volume is $2\pi^2 r^2 R$ using the **method of cylindrical shells**.



6. Use Simpson's rule with $n = 4$ to approximate the integral $\int_0^{\pi} \sin x \, dx$.

7. Evaluate the improper integral
$$\int_0^4 \frac{\ln x}{\sqrt{x}} \, dx.$$

8. Use the Comparison Theorem to determine whether the integral

$$\int_1^\infty \frac{x^3}{x^5+2}\,dx$$

is convergent or divergent.

9. For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}dx$$

convergent?

10. Determine whether the series is convergent of divergent. If convergent, find its sum.

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$$

Answers:

Only answers are provided here. You are expected to provide fully worked out solutions. If you need help with solving any of these problems please visit the Calculus Workshop or the Discussion board forum in Canvas.

1. (a) $-\frac{\pi}{6}+\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{3}e^{-x^3}(x^3+1)+C$ (c) $-\ln(|\sin x|)+\ln(|\sin x-1|)+C$ (d) $\frac{1}{2}(\ln|\tan x|)^2+C$
2. (a) T (b) T (c) T (d) T (e) F
3. use the arc length formula.
4. area is $\frac{3\pi a^2}{8}$, arc length is $6a$.
5. The diagram shows the torus can be thought of a solid of revolution. Determine the equation of the curve being revolved, and then evaluate the appropriate integrals representing volume.
6. $\frac{\pi}{6}\left(1+2\sqrt{2}\right)$
7. $8(\ln 2-1)$
8. Convergent.
9. Convergent if $p>1$, divergent if $p\leq 1$ (Note: we will use this result a fair bit in Part 5: Infinite Sequences and Series)
10. Convergent. The series sums to $\dfrac{e}{e-1}$.

7.3 Final Exam Practice Questions

The final exam may test on all material covered from the beginning of the semester up to and including material corresponding to Lecture 6.3.

We have covered quite a bit of material this term. This can be a little overwhelming so the following list is intended to give you an idea of some of the things you are expected you to know. This list is by no means exhaustive, but it does highlight some common questions I have been asked by a few students over the term.

You are expected to know:

- the **antiderivatives** in the table of Lecture 3.5 (page 89) - some of these can be derived from others in the table, and some can be derived using techniques we learned this semester. It is up to you to determine how many you should commit to memory (probably first 5 rows should be at your fingertips ready to be used in a pinch).
- all the various **techniques for integration**: substitution, by-parts, trigonometric substitution, partial fractions (Chapter 3)
- the **trigonometric identities**:

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

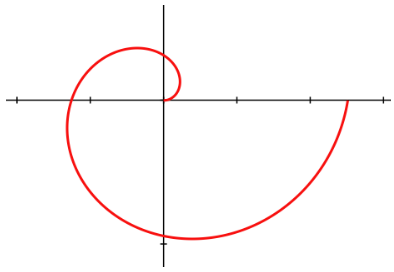
- the **Midpoint, Trapezoid and Simpson's Rule** and how to apply them. You will not be required to memorize the error bounds. But you may need to know how to use them, in which case they will be provided in the question.
- the formulas for **arc length** (Lecture 4.1), surface area (Lecture 4.2)
- **area** and **arc length** for curves given by **parametric equations or polar coordinates** (Lectures 2.2 and 4.3)
- **area of a surface of revolution** for a curve given by parametric equations (Lecture 4.3)
- **Series Tests**: integral test, comparison test, limit comparison test, alternating series test, ratio test, root test (Lectures 5.2 - 5.6) (See chart in Lecture 5.7 for the "big picture".)
- techniques for **approximating series**
- the definition of **Taylor Series and Maclaurin Series** and how to compute these series for a given function (Lecture 5.10)
- **Differential equations** (Lectures 6.1,6.2, and 6.3.)

Make sure you review ALL the questions from the homework assignments and midterms. It is expected that you will know how to do all of these at the time of the final exam.

The following questions can be thought of as a sample exam. The best way to use this resource is to sit down in a distraction free place, put your notes and textbook away, and then pretend you are actually writing the exam. Once finished, you should have an idea of what you still need to practice.

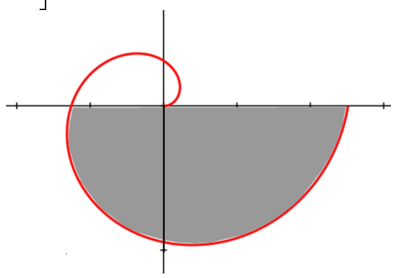
Solutions for the following questions will not be posted, though answers are included on the last page. If you want to determine whether you have done a question correctly then you can visit the TA's or myself in the workshop.

1. Compute the following integrals.
   (a) $\displaystyle\int_2^6 \frac{x+1}{x^2+2x-3}\,dx$
   (b) $\displaystyle\int \frac{e^x}{e^{2x}+2e^x+2}\,dx$
   (c) $\displaystyle\int e^{\sqrt[3]{t}}\,dt$
   (d) $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$
2. Find the solution to the differential equation that satisfies the initial condition:
$$\frac{dy}{dx}=\frac{y\cos x}{1+y^2},\quad y(0)=1.$$
   You may leave the solution in implicit form.
3. Find the length of the curve
$$y=\int_1^x \sqrt{t^2-1}\,dt,\quad 1\le x\le 4.$$
4. (a) The curve in the figure is called an Archimedean spiral and it has parametrization
$$x=t\cos t,\quad y=t\sin t\quad 0\le t\le 2\pi.$$
   Compute the length of the curve.

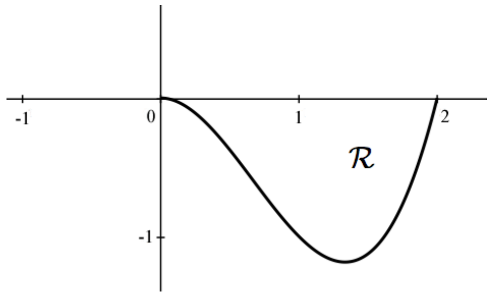


   Note: You may use $\displaystyle\int \sqrt{t^2+1}\,dt=\frac{1}{2}\left[t\sqrt{t^2+1}+\ln(t+\sqrt{t^2+1})\right]$.

   (b) Set up, but do not evaluate, the integral for the area of the region bounded by the Archimedean spiral and the x-axis as shaded in the figure.



   (c) Set up, but do not evaluate, the integral for the area of the surface of revolution obtained by rotating the part of the spiral in the second quadrant about the x-axis.
5. (a) Let $\mathscr{R}$ denote the region bounded by the curve
$$y=x^2(x-2)\qquad(0\le x\le 2)$$
   and the x-axis. Using the method of cylindrical shells compute an exact value for the volume of the solid of revolution obtained by rotating $\mathscr{R}$ about the y-axis. Draw a typical cylindrical shell in the diagram.



   (b) Formulate, but do not evaluate, an integral representing the volume of the solid of revolution obtained by rotating $\mathscr{R}$ about $x=3$. Sketch a diagram and indicate which method you are using: Washer Method or Shell Method.
6. Find the values of r for which $y=e^{rt}$ satisfies the differential equation
$$y''-y'-6y=0.$$

7. The population of a species of elk on Read Island in Canada has been monitored for some years. When the population was 600, the relative birth rate was found to be 35% and the relative death rate was 15%. As the population grew to 800, the corresponding figures were 30% and 20%. The island is isolated so there is no hunting or migration.
   (a) Write a differential equation to model the population as a function of time. Assume that relative growth rate is a linear function of population (i.e. use a logistic growth model).
   (b) Find the equilibrium size of this population.
   (c) Today there are 900 elk on Read Island. Oil has been discovered on a neighbouring island and the oil company wants to move 450 elk of the same species to Read Island. What effect would this move have on the elk population on Read Island in the future.
8. For each series determine whether if is convergent or divergent. In each case, state the test(s) you are using, justify the steps in using the test, and clearly indicate whether the series is convergent or divergent.
   (a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+1}$
   (b) $\sum_{n=1}^{\infty} \frac{n!}{2^n(n+2)!}$
   (c) $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$
   (d) $\sum_{n=1}^{\infty} a_n$ where a_n are defined recursively by the equations $a_1 = 2, \quad a_{n+1} = \frac{2^n}{n!} a_n$.
9. For each of the following power series compute the radius R of convergence and the interval I of convergence. Justify your answer.
   (a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 7^n}$
   (b) $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$
10. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 7$. What can be said about the convergence or divergence of the following series?

$$\text{(i)} \quad \sum_{n=0}^{\infty} c_n (3)^n \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} c_n (8)^n \qquad \text{(iii)} \quad \sum_{n=0}^{\infty} c_n (-5)^n$$

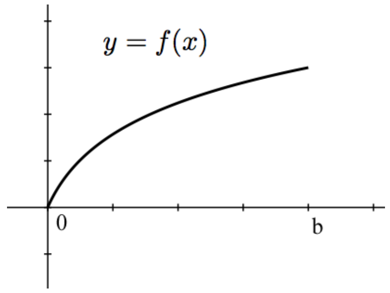
11. Prove that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

    is convergent for any $p > 1$.
12. (a) Define the **Taylor Series** of a function f at a.
    (b) Compute the Taylor series for the function $f(x) = \cos(3x)$ at $a = \pi/2$.
    (c) If the degree 4 Taylor polynomial $T_4(x)$ for $f(x) = \cos(3x)$ at $a = \pi/2$ is used to approximate f on the interval $\pi/4 \le x \le 3\pi/4$, use Taylor's Inequality to estimate the size of the error. (You may leave your answer in calculator-ready form.)

13. Suppose you want to approximate the integral $\int_0^b f(x)\, dx$ where f is the function given in the figure. You decide to use the Riemann Sum with 4 subdivisions (i.e. $n = 4$), but are debating whether to use a right-hand approximation R_4 or the Trapezoid rule T_4. Which

approximation will be more accurate? Give an explanation supporting your answer and sketch the approximating areas (rectangles or trapezoids) in the diagram.



Answers:

Only answers are provided here. You are expected to provide fully worked out solutions. If you need help with solving any of these problems please visit the Calculus Workshop.

1. (a) $\ln 3$ (b) $\arctan(e^x+1)+C$ (c) $3e^{3\sqrt{t}}(t^{2/3}-2t^{1/3}+2)+C$
   (d) $\sqrt{x^2-1}\ln x-\sqrt{x^2-1}+\arctan(\sqrt{x^2+1})+C$
2. $\ln|y|+\frac{1}{2}y^2=\sin x+\frac{1}{2}$
3. $\frac{15}{2}$
4. (a) $\pi\sqrt{4\pi^2+1}+\frac{1}{2}\ln(2\pi+\sqrt{4\pi^2+1})$ (b) $\displaystyle\int_{\pi}^{2\pi}-t\sin t(\cos t-t\sin t)\,dt$ (c) $\displaystyle\int_{\pi/2}^{\pi}2\pi t\sin t\sqrt{1+t^2}\,dt$
5. (a) $\frac{16\pi}{5}$ (b) shell method: $-2\pi\displaystyle\int_0^2 x^2(3-x)(x-2)\,dx$
6. $r=-2$ and $r=3$
7. (a) $\dfrac{dP}{dt}=\frac{1}{2}P\left(1-\frac{P}{1000}\right)$ (b) 1000 elk (c) A population of 1450 is larger than the carrying capacity of 1000 so $\dfrac{dP}{dt}<0$ which means the population would decrease back towards 1000 as time moved on.
8. (a) converges (b) converges (c) converges (d) converges
9. (a) $R=7, I=[-7,7]$ (b) $R=\infty, I=(-\infty,\infty)$
10. (i) converges (ii) diverges (iii) not enough information is given to decide convergence
12. (a) See definition in section 11.10 of textbook. (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$
    (c) $|R_4(x)|\leq\frac{3^5}{(4+1)!}(\pi/4)^{4+1}=\frac{3^5}{5!}(\pi/4)^5$ for $\pi/4\leq x\leq 3\pi/4$.

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